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THE

ELEMENTS

OF

EUCLID,

WITH

DISSERTATIONS,

INTENDED

To aſſiſt and encourage a critical examination of theſe Elements, as the moſt effectual means of eſtabliſhing a juſter taſte upon mathematical ſubjects, than that which at preſent prevails.

VOL. I.

BY JAMES WILLIAMSON, M.A.
FELLOW OF HERTFORD COLLEGE.

Sed nil dulcius eſt, bene quam munita tenere
Edita doctrina Sapientum templa ſerena. LUCR.

OXFORD:
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MDCCLXXXI.

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BIBLIOTHECA BODLEIANA

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TO

RICHARD BURKE, Esq.

Dear Sir,

IT would be a very entertaining ſpeculation, and well calculated to ſhew the importance of geometry, to enquire into the probable ſtate of human affairs at preſent, upon a ſuppoſition that the mathematical ſciences had been generally known and cultivated in the early ages of the world.

If they had only underſtood how to make a *map* or an *almanack*, a great deal of rational curioſity would have been gratified, which we now endeavour in vain to ſatisfy. For mankind have ſlept on, from age to age, without taking notice of time or occurences; and as to what concerns *geography* they have been equally remiſs. The human race have migrated from one country to another, until they have covered the face of the whole Earth, without gaining the leaſt information by their travels; having rather ſtrayed like cattle, than changed their habitation like rational creatures.

However the happy effects of a proper cultivation of this ſcience, are not confined to *maps* and *almanaks*: for it is by the judicious application of mathematical knowledge, that human nature, notwithſtanding its frailty, is rendered ſuperiour to every obſtacle, for our efforts keep pace with all our rational imaginations: and thus

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it riſes greatly above all its imperfections, acquiring a dignity which muſt aſtoniſh every one, when he compares what has been done, with the weakneſs and imperfection of any ſingle individual. And what this might have produced through a length of ages, operating in every quarter of the globe is not ſo eaſily to be conceived.

It has indeed been alledged, that the improvements aſcribed to the cultivation of this ſcience, have been often the effect of chance, and not produced by any rational ſcheme of improvement, conducted upon ſcientific principles. But the contrary appears from undoubted matter of fact; becauſe we every where find that diſcoveries are made among the moſt enlightened nations, and never among barbarians. Events, no doubt, will happen, and natural appearances will keep their regular time; but the wild beaſts of the field are as likely to make a proper uſe of them, as illiterate ſavages. And the little attention paid by ignorant men to thoſe very circumſtances which inform and enlarge the underſtanding, ſhew us better than any thing elſe, what human nature can riſe and fall to.

Both you and I are ſufficiently ſenſible of the great utility of mathematics in all the affairs of life; and that thoſe whom it concerns are, to a certain degree at leaſt, ſufficiently attentive to it. And we only uſed to regret that the world is ignorant, how ſucceſsfully this ſcience might be employed for making us rational creatures.

It has often been the ſubject of our converſation and ſurprize to run over the various methods contrived for making mathematicians, without impoſing upon them the neceſſity of being rational creatures. And ſuch diſcourſes have generally ended in your urging me to the execution of this plan; the firſt part of which I here beg leave to preſent to you. I have had ſufficient marks of your friendſhip to be ſenſible that you will be much more ſollicitous about its ſucceſs than I am myſelf: and from your partiality to me, I am alſo perſuaded you will think that I have not done it juſtice in the execution.

It is true I could have improved the ſtyle very much; but it ſeems to anſwer my purpoſe better in its preſent form: for I write not to make people read, but to make them think. I have alſo affected a familiarity of phraſe, to engage the attention of the reader by expreſſing geometrical ideas in common language.

But whatever your opinion of the work may be, I beg of you to accept of this addreſs, as a teſtimony of the high eſteem which I have for your character and abilities. It is with the greateſt pleaſure that I recollect the ſhare which I have had in your education, which was carried on, not in the perſon of maſter and ſcholar; but rather in the character of two friends who had taken a ſomewhat different view of the ſame ſubject.

I am, DEAR SIR,

with the greateſt reſpect,

Your moſt obedient

humble ſervant,

HERTFORD COLLEGE:
March 27. 1781.

JAMES WILLIAMSON.

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DISSERTATION I.

IT is my intention in this diſſertation, after ſaying ſomething concerning the uſe of commentaries, to conduct the learner, ſtep by ſtep, to a ſolid, rational and extenſive view of the principles of geometry; by explaining their original and improvement through ſeveral different gradations, until they put on a ſcientific form; reducing what I have to ſay under diſtinct heads, for the benefit of the ignorant and thoughtleſs reader.

CHAP. I.

Of the neceſſity and uſe of commentaries.

THE dulneſs of commentators is a ſubject of ſuch general complaint, that it may be proper to inquire how it comes to paſs that books are not written in ſuch a manner as to make a commentary uſeleſs or impertinent: and this inquiry may be propoſed with the greater propriety, as the labours of commentators, beſides their dulneſs, ſeldom or never anſwer the expectations of the public in other reſpects; which they no doubt are apt to imagine the author himſelf could have fully gratified, by condeſcending a little to the weakneſs of his readers, and reſcued his works out of ſuch clumſy hands.

What apology therefore ſhall I be able to make for loading with a commentary the moſt perfect book in the world! But though it may not be agreeable to that general delicacy which an author is bound to preſerve towards his readers; yet it may nevertheleſs be proper to inform them, that they ſeem to be rather partial to themſelves upon the preſent queſtion; never conſidering how much of the blame ought in all conſcience to be laid at their own door: for though it may be a ſecret to a great many, yet it is an undoubted truth that the moſt perfect writer requires that his work ſhould be put into the hands of thoſe who can read. And my intention in this edition is not to correct my author, but to ſupply a defect which it could not have been very conſiſtent with his plan to remedy. For he has written his book expreſſly upon the ſuppoſition that his reader was endued with the faculty of attention: and as this is a diſpoſition of mind with which the book is but rarely taken up, though it will always be laid down with it, or elſe it has been uſed to little purpoſe: a few ſeaſonable warnings therefore, to rouſe the attention of the indolent reader, may be given with great propriety, and without bringing any reflexion upon the character of the author, who in point of perſpicuity, excells every one.

But I will even go farther and venture to affirm that an author, who writes upon ſubjects of ſcience, may find it often by no means convenient to deliver himſelf in ſuch a manner as to be always intelligible even to thoſe whom he would wiſh to have for readers. Becauſe authors are confined to a particular method of arrangement, if it be their intention to deliver opinions or diſcoveries in a ſyſtematic manner. And although there has been a ſenſeleſs and inceſſant clamour, for almoſt two centuries, againſt this method; yet it ſeems to be the only way of keeping knowledge within ſuch limits, as to be by any means manageable by the human mind.

It is true indeed that our progreſs in acquiring knowledge, when left to ourſelves, and our own experience, is directly contrary to this, being firſt condemned to an examination of particulars. And even when we take up with the ſyſtems of others, it is requiſite that we have laid in a ſufficient ſtock of materials to enable us readily to comprehend any very general doctrine. For no general rule, or law, or theorem, or what you pleaſe to call it, is by any means to be underſtood, unleſs we have particular inſtances ready to apply, as occaſion requires: always however excepting a *genius* of the kind given to *Hudibras* in the following lines, containing the meaning and importance of many volumes.

His notions fitted things ſo well,
That which was which he could not tell.

The fault therefore is not in the ſyſtems, but ariſes from the general incapacity which mankind ſeem to labour under, for judging of the merits of ſuch as have been offered to their conſideration; and as there are quacks of all denominations, ever lying in wait to take advantage of the ſimplicity of the multitude; the world by this means has been over-run with counterfeits.

To have a genius for any ſcience ſeems to me to imply a readineſs at finding out particular inſtances to apply to any general rule. It is therefore to be ſuſpected that thoſe who are deficient in this kind of invention will by no means find their progreſs in any ſcience, anſwerable to the time which they beſtow upon it: they may commit the rules or the theorems to memory and neverthelefs be ignorant of their meaning and application. Hence a certain degree of invention becomes neceſſary even for the ready acquiſition of a ſcience; and this perhaps not ſo different from that kind of invention by which the principles were at firſt diſcovered, as many have been apt to imagine.

Now here is the difficulty; a ſcientific book ought to contain knowledge in that compact form; in which every one would chuſe to take a review of it, after he has made himſelf maſter of the ſubject: and not incumbered with all thoſe particular inſtances, which would now ſtand in the way of the readers imagination as much as they aſſiſted it before. For inſtance, who could reliſh the nobleſt of Newton's theorems if it preſented itſelf to his mind encumbered with all thoſe properties, which had led him from common notions to the right underſtanding of ſo ſublime a ſpeculation.

A book therefore, if this reaſoning be juſt, which would be proper for a learner would be fit for nobody elſe: and the greateſt perfection of writing will ſtill leave occaſion and employement for the talents of that kind of commentator, which I profeſs myſelf to be. The ſweepings of the author's ſtudy would furniſh the beſt and moſt authentic materials for works of this kind; and yet humble as the office may ſeem, if I can execute my taſk but nearly up to the idea which I have of it, I ſhall not regret my labour. The reader however will be diſappointed, if he expect to find Euclid either corrected or enlarged, my purpoſe being nothing more than to conduct the learner to that ſenſe in which the author wiſhed himſelf to be underſtood. An oſtentation of learning is a fault which prevails among many commentators; for they ſeem to be apprehenſive leaſt the world ſhould think they know nothing but their buſineſs: and therefore inſtead of explaining their author, endeavour to perſuade the world that they underſtand the ſubject better than he himſelf did. But I do not write to the ſtrength but to the weakneſs of mankind; and therefore would not chuſe to be conſidered as challenging the whole world to find fault, but only as applying a remedy to a weakneſs which my own experience has found does exiſt.

CHAP. II.

Concerning the Original of the geometrical principles.

QUANTITY of both kinds, extenſion and number, is always forcing itſelf upon us whether we will or not; and muſt therefore leave ſome very fixt and determinate impreſſions upon the minds of the moſt inconſiderate. We cannot ſtretch out our hand without receiving a perception of extenſion; nor open our eyes without ſeeing figured objects, bringing along with them to the mind a conſciouſneſs that they are more in *number* than one. This begets notions, which are by no means peculiar to any ſingle art or profeſſion, but common to all men. The geometrician ſelects the moſt accurate of theſe, and with ſuch materials lays the foundation of his ſcience. Particular circumſtances have rendered ſome notions concerning quantity more invariable than others; or rather ſo fixt that nothing can alter them. As it would diſcover ridiculous affectation and ignorance to pretend to change theſe, which is indeed impoſſible, ſo it is alſo below the dignity of our author to affect to diſguiſe them by any forced or unnatural conſtruction, to make them wear a more philoſophic appearance. But although the twelve common notions which he has ſelected, are to be underſtood according to the vulgar conception of them; yet the learner muſt give them a very particular examination: becauſe it is not ſufficient to have theſe notions, but we muſt alſo have the ready uſe of them. And this may be acquired in ſome ſuch manner as the following.

Let the ſtudent provide himſelf with a ruler and compaſſes, and after ſome practice in drawing ſtraight lines, and deſcribing circles; he is next to proceed to the examination of the common notions, as if they were properties of ſtraight lines only, and true of nothing elſe. For without this precaution he will undoubtedly be liable to have the diſtich quoted in the laſt chapter applied to him. And any tincture of the *hudibraſtic genius* diſqualifies a man for this ſcience; and excludes him from a great deal of rational amuſement, to ſay nothing of more ſolid advantages. I ſhall therefore at the porch, not only lend the learner my advice but alſo my aſſiſtance in ſtriping himſelf of thoſe prejudices which would diſgrace his behaviour after he has been admitted into this magnificent temple where all the wonders of the world are diſplayed.

The reader may believe that I never would have introduced this advice with ſo much form and circumſtance, without a firm perſwaſion that it is of the laſt importance. He is therefore immediately to ſet about the work, by deſcribing a circle, not a geometrical but a mechanical circle; and ſuch as any ordinary compaſſes will exhibit; drawing at the ſame time ſeveral ſtraight lines from the center to the circumference. He is next to ſatiſfy himſelf of the equality of theſe ſtraight lines, by meaſuring them with his compaſſes: his concluſion will be, that they are equal; and he will find his opinion of their equality grounded upon the firſt common notion; becauſe they are all equal to the ſame length, viz. the diſtance between the extreme points of his compaſſes. But it is carefully to be obſerved that this is not to be made the ſubject of a tranſient reflexion, but of frequent and cloſe meditation; varying the center and radius to the utmoſt limits of the compaſſes; with now and then a thought upon the limited nature and imperfection of the inſtruments.

The ſecond and third of the common notions may be examined by deſcribing two circles with the ſame center, but at different diſtances, and drawing ſtraight lines from the center to the remoteſt circumference; the parts of the ſtraight lines intercepted between the two circumferences are equal; and will illuſtrate the ſecond common notion by taking the leſs radius from the greater. And thus we are to proceed untill we have ſatisfied ourſelves that theſe common notions are true at leaſt of ſuch ſtraight lines as we can draw upon a piece of paper.

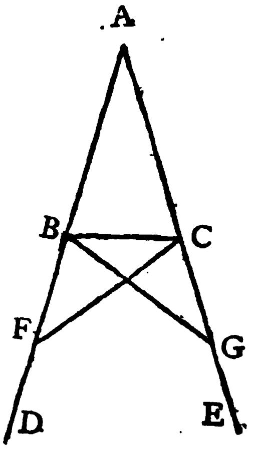
I beg the reader's pardon for my impertinence; but he is farther to be admoniſhed, that it is not ſufficient to run theſe things over in his own mind; but that he muſt be able to expreſs them to the conviction of a by ſtander; and this will make it neceſſary to diſtinguiſh his lines and circles by the letters of the alphabet.

CHAP. III.

The ſame ſubject continued.

SUPPOSING this buſineſs of the ſtraight lines accurately diſcuſſed; the learner is next to ſhut his compaſſes; and then obſerve their progreſs in opening until they take the direction of a ſtraight line: during this operation, he will find the inclination of the legs continually varying: at firſt nothing, then gradually increaſing until it diſappears when the legs become one ſtraight line. This inclination is a quantity, though not a tangible ſubſtance, but this the reader will do well to convince himſelf of; and for this purpoſe he may obſerve that any particular inclination may be equal to another, or the half or the third part of it. But the common notion of this kind of quantity is not ſo regular or determinate as that of a ſtraight line; though it exhibits every poſſible ſhape which it can take in opening the compaſſes as above directed: the reader therefore will be pleaſed to inſtruct himſelf properly in this: and then proceed to examine whether the common notions are not alſo true when applied to this kind of quantity.

And for this purpoſe I would recommend a triangular piece of wood, of the ſhape of a right angled triangle with unequal ſides, being afraid to meddle with circular arches, leaſt we ſhould *conjure up* a prejudice which we might want art afterwards to *lay*. By the aſſiſtance of this triangular piece of wood, make two equal inclinations (or angles) upon paper, taking care to make the lines unequal, to prevent prejudice. After theſe are made, their equality may be inferred from the firſt common notion, as each of them will be equal to the inclination of the two ſides of the peice of wood: add to theſe two equal angles, other two equal angles; which may be done by the aſſiſtance of a different corner of the ſame piece of wood; and this will illuſtrate the ſecond and third; according as you conſider one of them as taken away from the whole angle made up of the two; or as added together to make one. But it will be neceſſary previous to this, to acquire a ready and accurate way of expreſſing the different inclinations of lines, (called angles) by the letters of the alphabet. The figure annexed will be a very proper one for practice and the taſk which I would ſet the reader is to tell the number of angles and the different methods of expreſſing them; giving him to underſtand that their number is above fourteen; and that, CAB, CAF, CAD; GAB, GAF, GAD; EAB, EAF, EAD; BAC, BAG, BAE; FAC, FAG, FAE; DAC, DAG, DAE; are only ſo many different ways of expreſſing the ſame angle; nor does this great variety, in the leaſt puzzle or perplex the conceptions of an *adept*. This looks ſo much like a riddle that I think it cannot fail to engage the attention of the curious. But not to truſt entirely to the reader's own ingenuity for unraveling this knotty point; let him obſerve the following hints; the letter at the meeting of the lines, whoſe inclination to one another we want to expreſs, is put in the middle, and it is ſufficient that the other two letters, each expreſs ſome point in each line: thus the inclination of FB to BC is called the angle FBC or CBF: and the inclination of DB to BC is the very ſame with the other, as is obvious, and is called the angle DBC or CBD: the inclination of BC to CE is called the angle BCE or ECB; and the inclination of GC to CB is the ſame with the other and is called the angle GCB or BCG. But farther the angle ABG is made up of two angles viz. ABC, CBG: and the angle ACF is made up of two angles viz. ACB, BCF. And to aſſiſt the reader in applying the ſecond common notion I have made the angle ABG equal



equal to the angle ACF: and I have likewiſe made the angle CBG equal to the angle BCF; and the concluſion will be that the angle ABC is equal to ACB.

CHAP. IV.

In what manner our common notions begin to take a ſcientific form.

AFTER the reader has prepared himſelf according to the directions given in the laſt two chapters; it will now be proper to take a review of the inſtruments, which he made uſe of, for regulating his conceptions: and theſe, he will find, were very limited, being confined to a few inches. Let him next aſk himſelf, whether he has any reaſon to ſuſpect, that the concluſions, obtained by the help of theſe inſtruments, were equally limited. Nor is this point to be determined raſhly, but ſo as to be ſtill certain that he treads on firm ground: and we may venture to draw this concluſion for him; that, without any great force upon his imagination, he can conceive his inſtruments to have a double or triple extent without finding the leaſt reaſon to change his opinions. And by proceeding thus, he will certainly come to this concluſion at laſt; that although theſe inſtruments might be the occaſion of his turning his thoughts to this ſubject, yet his opinions were nevertheleſs derived from the nature of extenſion in general; and that they knew no other limits, but ſuch as bounded extenſion itſelf; but more particularly, that a circle whoſe radius is a thouſand miles, or the thouſand part of an inch, would furniſh the ſame concluſions as one of two or three inches. Here now our opinions, which before were meaſured by our inſtruments, begin to put on a different form and diſplay to our imagination the firſt dawn of ſcience.

If any one ſhould pretend that he had the notions orginally in this very general form to which I have been endeavouring to lead him; I have only to ſay, unleſs they were acquired by an examination of particulars, he will find his notions fit every thing ſo well, that when he comes to apply them to particular inſtances, he will not be able to tell *which is which*.

The reader is to endeavour next to get ſomething like a ſcientific notion of an angle, by correcting the vulgar notion of an angle, by which is underſtood the corner of any thing. Now this does not ſo much depend upon any ſtretch of the imagination, by which large objects, and ſuch as exceed the experience of our ſenſes, are to be made the ſubject of Contemplation; becauſe the point where the lines meet, together with any point in each of the lines fixes the angle invariably: or in other words, the three points denoted by the three letters of the alphabet, expreſſing the angle, fixes any rectilineal angle: for the angle is not changed by making the lines longer or ſhorter; but only by opening or ſhutting them; conceiving them to turn upon a pin like the two legs of a pair of compaſſes.

But our inſtruments are not only too limited for our conceptions, but are inaccurate in other reſpects. We have a very clear notion of three dimenſions viz. length, breadth and thickneſs: and ſurely without nicely ſeparating and diſtinguiſhing theſe, it is impoſſible to have true and proper conceptions of magnitude. But theſe different dimenſions cannot be repreſented by our inſtruments. For when we attempt to draw a line or even to mark a point; our line and point poſſeſs all the three dimenſions in as great perfection as a cannon ball or the maſt of a ſhip. The human mind, when once made ſenſible of its powers, will never ſuffer its conceptions to be ſo cloged with matter: which has put thoſe who carry their views beyond the vulgar, upon inventing ſome method by which our conceptions may be rendered more rational and conſiſtent; and this is the original of definitions.

CHAP. V.

Of definitions.

OUR author has proceeded with ſingular judgement in laying down his principles: where the common notions are ſufficiently diſtinct and accurate, he has inviolably adhered to them. But when theſe are too incorrect or too indeterminate, he explains the ſenſe in which he would have any particular term be underſtood; and what conception he requires his reader to have of the figures which he defines. Definitions may be conſidered as of two kinds; firſt, ſuch as ſerve only to explain the meaning of a word; but theſe are not properly geometrical definitions, for from this no conſequence can follow unleſs there be a myſtical virtue in the name. Secondly, thoſe from which conſequences or properties do follow, which may be called geometrical to diſtinguiſh them from the others. The definition of a circle, given by Euclid, is of this kind for all the properties of a circle follow as conſequences from it. And here the reader is to be admoniſhed, that, upon this and ſuch like occaſions, his common notion of a circle is to be laid aſide, and nothing admitted as a property of this figure, unleſs it can be ſhewn to follow from this definition.

Though Euclid in the arrangement of his principles has placed the common notions after the definitions, yet they are prior to them in the order of conception; and indeed if this is not attended to, ſome of his definitions will be unintelligible; for inſtance his definition of a ſtraight line. He ſays a ſtraight line is that which lies evenly to the points in itſelf. Now if I am to conceive nothing previous to this, reſpecting a ſtraight line; what can I underſtand by this definition; or what can I infer from it? The reader will be juſt as much at a loſs to conceive the meaning of the word *evenly*, as of the ſtraight line itſelf. But if we conſider this definition as an improvement upon the common notion of a ſtraight line, (ſee com. not. 12.) every thing is very intelligible: for after a proper examination of this principle, that two ſtraight cannot incloſe a ſpace; every body will infer, though not ſcientifically nevertheleſs very confidentally, that every ſtraight line muſt lie evenly to all the points in itſelf; otherwiſe he certainly might have hopes at leaſt of making two of them incloſe a ſpace. I would be rightly underſtood upon this point; nobody can imagine that it is my opinion, that Euclid intended that the one of theſe ſhould be inferred from the other ſcientifically; but only that the definition expreſſes the conception, derived from two lines, reduced into a more ſimple form; though indeed he himſelf reaſons from the common notion as will appear in the fourth propoſition.

Dr. Hooke one of the greateſt improvers of practical mathematics, was notwithſtanding ſo ſenſible of the neceſſity of laying an accurate foundation in theory that he thus ſpeaks concerning a point. *A point is that which has no part.* " This which ſome

" would deem the moſt inconſiderable thing in the world, ſeems " yet the moſt difficult to be underſtood; no ſenſe, or imagination " or fant'cy can reach it, nor words deſcribe it, but by a negative, " to tell you what it is not: For it is not to be taken in the ſenſe, " that the whole Earth is called a point in reſpect of the Univerſe, " nor in the ſenſe that the end of a tapering thing is called a " point, as of a pin or needle; though they ſeem to be the ſmalleſt " things we know; becauſe theſe latter may be ſaid to have as " many parts as the fore-mentioned; for ſince all quantity is divi- " ſible *in infinitum*, the leaſt quantity may be divided as often as " the greateſt, and therefore whatſoever is diviſible muſt have " parts, and therefore none of theſe can be properly called a point, " in the ſenſe here named, unleſs this point be underſtood to be " the *apex* of a mathematical cone or pyramid, where the ſuper- " ficies of it is determined, for that will be a mathematical point: " But it cannot be ſuppoſed of a phyſical point, or material cone " or pyramid, for that will have extenſion and bluntneſs. And we " find that *microſcopes* will make thoſe points diviſibe even to ſenſe, " nay even almoſt to diſcover a new world in a point, nay there is " one now that affirms he has ſeen more than ten thouſand living " creatures in the bigneſs of a very ſmall ſand, which itſelf indeed " is but a viſible point to the naked eye; and each of thoſe ten " thouſand may have worlds within them. We know not the " limits of quantity, matter, and body as to its diviſibility or ex- " tenſion; no imagination can comprehend the *maximum* or the " *minimum naturæ*, our faculties are finite and limited, and we " muſt content ourſelves within the orb and ſphere of their acti- " vity. And acquieſe in a negative definition, and underſtand if " we can ſomewhat that is ſmaller than the ſmalleſt, though that " be alſo improper; for in that which is not quantity, there is " neither ſmaller nor bigger, we muſt endeavour to underſtand " ſome what infinitely little, leſs than which there cannot be, " ſomewhat that has no bigneſs or extenſion or quantity, but only " poſition and reſpect to quantities circumjacent: From which, to " this or that body, there is a determinate length and diſtance; " and upon this account, wherever we endeavour to underſtand " this notion, our imagination will repreſent to us the ſmalleſt

" viſible body, as an exceeding fine ſand, or a mite, or the point " of a needle, or the ſmalleſt viſible body we have ever ſeen on " paper, or the like; which we muſt be content with, ſince the " fant'cy forms nothing but what is firſt in the ſenſe, though it be " none of theſe. And in truth it can have no true definition that " will reach its eſſence. Analogous to this *point*, ſign or nothing " in quantity is the *nought*, cifer or zero in number: The *never* " in time: The reſt or quiet in motion. For as no aggregate of " points will ever produce a line or quantity; ſo the multiplication " of *noughts* or *cifers* will never produce a number; and as the " addition of *nevers* cannot make time, ſo the aggregate of *reſts* " cannot produce motion. So that all theſe may not improperly " be called the *terminus* or bound, from which they all begin; ſo " quantity may be ſaid to begin from a point or nothing: number " may be ſaid to begin from *nought*, cifer or zero: time may be " ſaid to begin from *never*: and motion to begin from reſt. And " as the *minimum naturæ* may be ſaid to be the firſt quantity; if at " leaſt there be a *minimum* in nature, ſo an unite may expreſs it in " numbers; *inſtant* in time, and moment in velocity. It may poſſibly be thought I have ſaid too much of *nothing*, but yet it " ſeems to be of the greateſt conſideration in nature; for it ſeems " to be the beginning of every creature; even the greateſt creatures " having been traced to begin from an atom or point, no eye or " ſenſe can reach it; nor any underſtanding limit it; that the beginning of a very large animal hath been ſeen alive, ten thouſand " times ſmaller than a mite, may be proved, and yet how much " ſmaller it may have been is not determined."

It has been ſaid of the muſick of the ſpheres that it is ſo loud we cannot hear it: and of a mathematical point, according to this explanation, it may be ſaid, that it is ſo ſimple it is impoſſible to comprehend it. This no doubt, muſt prepoſſes the reader with ſtrange apprehenſions of difficulty in the proſecution of a ſcience which ſets out ſo unaccountably; requiring us to admit as a principle what we cannot comprehend.

I would not have thrown this metaphyſical bugbear in the way of the reader; only I was afraid leſt he might ſtumble upon it of himſelf, with ſome hazard to his underſtanding, if he ſhould purſue the metaphyſical analyſis of it, until he had no doubts left but that it was really incomprehenſible; eſpecially if he happened at the ſame time to diſcover, that the ſame ingenious ſubtilties would alſo apply to lines and ſurfaces, by which means they might likewiſe be entitled to a place among the incomprehenſibles.

I ſhall therefore beg leave to call his attention to a method of explanation very different from this, but which ariſes very naturally from what I mentioned before viz. that the definitions ſhould be conſidered as ſubſequent to the common notions; and introduced merely as auxiliaries to them. And in the preſent inſtance, we have a very clear notion of three dimenſions in magnitude; though we always find them ſo connected that it is impoſſible to ſeparate them from one another. Let us therefore try to contemplate them together, but one after another. Every part of ſpace that we conſider has a ſhape; for inſtance the ſhape of a room: this ſpace the mathematicians call a ſolid; that which limits it or gives it this ſhape is called a ſurface; which muſt be conſidered as having only length and breadth; becauſe if it had thickneſs alſo; then it would not be the boundary of the ſolid, but a part of the ſolid itſelf. Therefore the proper definition of a ſurface will be, that which hath length and breadth. But farther this ſurface has a ſhape or is limited; that which limits it cannot have length and breadth; for then it would not be the limit of the ſurface but a part of it. Therefore the proper definition of a line is length without breadth. Laſtly this line has its limits; which limit cannot have length, for then it would not be the limit of the line but a part of it. Therefore the proper definition of a point would be, that which belongs to magnitude, and has no parts, that is none of the common dimenſions, length breadth or thickneſs. Thus it is eaſy to conceive how theſe definitions of a point line and ſurface are derived from the common notions of magnitude.

CHAP. VI.

The ſame ſubject continued.

AS the definitions which we have been explaining are never quoted or appealed to as teſts by Euclid, the learner might be ready to conclude that they were placed here for no other purpoſe but to impreſs one with an idea of the difficulty of this ſcience.

Though it be true that they are never uſed directly, yet by being placed where they are, they ſtop the mouths of impertinent critics, who would be ready to ſtart objections which are immediately removed by theſe definitions.

In this chapter I intend to explain what is meant by a PLANE, a plane rectilineal angle, a right angle and an acute and obtuſe angle; firſt conſidering what opinion every one might form of theſe, by examining thoſe inſtruments or circumſtances which would fix his attention to this ſubject.

To underſtand rightly Euclid's definition of a plane ſurface, it will be neceſſary to have recourſe to the former ſuppoſition, that the definitions are improvements upon our common notions: for inſtance, let us ſuppoſe one at work, in the manner of a carpenter, to ſatisfy himſelf that a ſurface was even, ſuppoſe the ſurface of a table: he would apply a ruler in all directions from ſide to ſide; and finding that it touched the table in every point and in every direction, he would conclude that the ſurface was even. And now we ſhall find that the definition is nothing elſe, but giving the notion acquired by this operation, a regular form and ſufficient extent,

For Euclid's plane ſurface which is commonly called a PLANE is this ſurface of the table, endued with a ſcientifical evenneſs inſtead of a mechanical one; and inſtead of being confined to any ſhape; of an unlimited extent. It is *in* or *upon* ſuch a ſurface as this, that Eulid ſuppoſes all his lines and figures to be deſcribed and drawn, in his firſt ſix books: and this is the more carefully to be obſerved, becauſe, though it is not mentioned in the demonſtrations, yet there are many properties of lines and figures, which are not true, but upon the ſuppoſition that the lines are in the ſame plane, as I ſhall have occaſion to obſerve.

And the plane rectilineal angle is the inclination of two ſtraight lines to one another *in* or *upon* ſuch an even ſurface as this, which meet together but are not in the ſame direction. They might be inclined to one another ſo that when produced they would interſect or croſs one another; but until they have met they are never conſidered as containing an angle.

This plane rectilineal angle may be of three different kinds; according to the different inclinations which two ſtraight lines may have to one another. Take two pens, and run a pin through the extremity of the one, and at the ſame time, through the other, at ſome diſtance from its extremity, ſo that the one of the pens may make two angles with the other, on each ſide one. Turn the pens into the ſame direction, and making one of them revolve, and then obſerve its progreſs in revolving until it takes the ſame direction on the other ſide: and during this whole revolution there is but one fixed or determinate poſition of the lines with reſpect to one another; and that is when the revolving line has the ſame inclination to the other on each ſide; and theſe equal inclinations are called right angles; and the other two have their names according as the angle is greater or leſs than this.

The reader has been very inattentive to my directions upon this head, if he does not perceive, that there is an unlimited variety both of acute and obtuſe angles, but that there is but one right angle.

And hence by the bye it is obvious that Euclid formed his definitions from common notions, as he reckons this one; that all right angles are equal, probably as a hint to the reader, that he might diſcover from what ſource the definitions were derived; becauſe this common notion may be regularly demonſtrated from the definition of a right angle, and therefore muſt have been placed where it is for ſome indirect purpoſe. Upon the whole then we may conclude, that Euclid's definitions are not the imaginary phantoms which the metaphyſicians have repreſented them: but notions derived from material objects, and new modeled, not for the purpoſe of empty ſpeculations, but, that they might be applied to the ſame kind of objects which firſt ſuggeſted them, upon a more accurate and enlarged plan.

CHAP. VII.

Of poſtulates.

THIS ſcience has now acquired ſuch an extent and accuracy, by improveing the common notions into definitions; that our former inſtruments are now become inadequate to our purpoſes and views. For our notions both of a ſtraight line and circle are ſo refined and enlarged; that no inſtrument is ſufficiently accurate or extenſive for performing theſe two problems, the drawing a ſtraight line and the deſcription of a circle.

Here the caſe of the ſcience appears deſperate; and the author reduced to the neceſſity of giving up his high pretenſions to accuracy and univerſality, by ſuffering the ſcience to fall back again into its original ſtate, as far as reſpects its inſtruments at leaſt. And I am certain the artifice uſed upon this occaſion to preſerve the dignity of the ſcience is by no means generally underſtood; and this will be confirmed beyond contradiction by attending to the fooliſh and childiſh reaſons generally given for admitting theſe poſtulates, alledging that it is ſo eaſy to conceive how they may be done, that they are to be admitted without the leaſt ſcruple or heſitation. That a thing may be eaſily done is ſurely a very bad reaſon for neglecting to give rules for doing it: and if that had been the caſe in the preſent inſtance, I am pretty confident theſe commentators would have been ſaved the trouble of their apology. But the opinion of Newton I ſuppoſe will be deciſive upon this occaſion, and we find him expreſſing himſelf directly to our purpoſe as follows. "Nam et linearum rectarum et circulorum deſcriptiones, in quibus *geometria* fundatur, ad *mechanicam* pertinent. Has lineas deſcribere *geometria* non docet ſed poſtulat. Poſtulat enim ut tyro eaſdem accurate deſcribere prius didiceret, quam limen attingat *geometriæ*; dein, quomodo per has operationes problemata ſolvantur, docet; rectas et circulos deſcribere problemata ſunt, ſed non geometrica. Ex *mechanica* poſtulatur horum ſolutio, in *geometria* docetur ſolutorum uſus. Ac gloriatur *geometria* quod, tam paucis principiis aliunde petitis, tam multa præſtet.

After ſuch an authority the judicious reader will be ready to agree with me, that theſe problems are taken for granted, not becauſe

cauſe it is eaſy to conceive how they may be done, but for a much better reaſon, becauſe it is impoſſible to do them by any method conſiſtent with the principles of this ſcience.

But even in this, which muſt have been a matter of no ſmall difficulty, our author has done every thing in the power of man, to render this unpromiſſing part as correct as poſſible, by making it put on a very reſpectable figure indeed, giving it ſuch an air, as diſcovers but little of its mechanic original,

An ordinary genius would probably have given directions for drawing a ſtraight line and deſcribing a circle, with a ruler and compaſſes; and when the limited nature of his inſtruments was mentioned as an objection to his ſcience, and that he could not with propriety extend his concluſions beyond a ſheet of paper; he would then probably propoſe a chain and poles by the aſſiſtance of which he might undertake to enlarge the ſcale of his operations; yet with waggon loads of ſuch inſtruments he never could hope to produce one ſcientific conſtruction.

But Euclid has acted with more judgement, and never would be acceſſory to the opening a door for admitting upon his ſcientific ſtage the whole tribe of mechanics, with the ſeveral implements of their trade. He throws them two problems to perform, but this is to be done behind the ſcenes, without exhibiting their mechanical inſtruments, declaring that upon every future emergency, he is determined to ſtand or fall by his own principles.

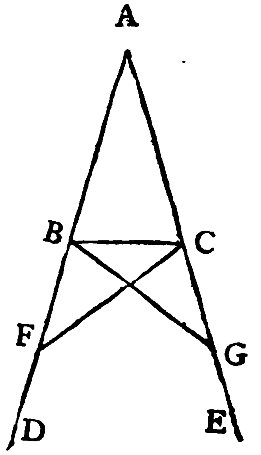
CHAP. VIII.

Of the inſtruments made uſe of for communicating geometrical knowledge.

ALTHOUGH language be a general medium for communicating knowledge of every kind; yet particular ſubjects require the introduction of ſome auxiliary inſtruments. Unleſs the nature of the thing is ſuch, that it cannot fail of itſelf to make a ſtrong impreſſion upon the mind, or that it is of no great conſequence whether it be particularly examined or not, it will be found difficult to command the attention; and particularly, which is abſolutely neceſſary in a chain of reaſoning, to make the thoughts of the reader and writer follow each other in the ſame order, with a certainty that they are both thinking of the ſame thing at the ſame time. The art of logic is an inſtrument of this kind, but the great diverſity of opinions prevailing among mankind is a ſufficient proof that this inſtrument is either neglected or miſapplied. Indeed that wonderful uniformity of opinion which has been univerſally found among all nations, and in all the different ages of the world upon geometrical ſubjects, cannot fail to draw our attention to this ſingular circumſtance; and make the learner deſirous to be let into the ſecret of this harmony, as far as one ignorant of the ſubject can be ſuppoſed capable of entering. And this will be found to proceed from the method of demonſtration to which geometricans have tied themſelves down; and their being poſſeſſed of an inſtrument, perfectly adequate to the purpoſe of communicating their knowledge to others, and by which they are able infallibly to examine their own. And this inſtrument in geometry is figures or diagrams: which Euclid expreſſes by the different names of σχῆμα, καταγραφη, εἶδος. The firſt is his general term, the ſecond he commonly applies to a figure which has been deſcribed or at leaſt compleated; and the third commonly to ſimilar figures.

Now theſe diagrams or figures may be made ſo exactly to reſemble the ſubject matter of any propoſition, that, if we think at all, it is impoſſible to miſtake the order of thinking, which the author has preſcribed, or to draw a different concluſion from that, to which he intends to lead us. I ſay a different; becauſe the figure itſelf may lead us to a limited or partial concluſion; the remedy for which I ſhall explain at length in the next diſſertation.

The ſimpleſt kind of rectilineal figure is the triangle; the different parts of which the learner ought to be familiarly acquainted with; and to keep in his mind, that beſides the triangular ſpace itſelf; there are ſix different magnitudes which go to the making it up; viz. three ſides and three angles; each of which ought to be particularly attended to. It will be uſeful allſo to obſerve; that the triangles ABG, ACF though they have ſeveral parts in common, are nevertheleſs to be conſidered as much as diſtinct and different triangles, as if ſeparated from each other by the diſtance of a thouſand miles.



The three ſides of the triangle ABG are AG, AB, BG and the three angles are ABG, AGB, GAB: and of the triangle ACF the three ſides are AF, AC, CF; and the three angles ACF, AFC, CAF and the angle at A or DAE is ſaid to be common to the two triangles; and therefore they have one angle equal to one angle; and it is ſaid to be contained by the ſides FA, AC when it is conſidered as an angle of the triangle ACF: but when it is conſidered as an angle of the triangle ABG it is ſaid to be contained by GA, AB.

Again FBC and BCG are alſo two diſtinct triangles: and the three ſides of the triangle FBC are BF, FC, CB and the three angles BFC, FCB, CBF: and the three ſides of the triangle BCG are CG, GB, BC; and the three angles CGB, GBC, BCG: theſe triangles have a common ſide viz. the ſide BC; and this ſide is ſaid to be extended under the angles BFC and BGC. Theſe things ought to be well underſtood and ſtrongly impreſſed upon the memory.

But farther, we find that the two triangles ACF and BFC have a ſide and an angle common to both, viz. the ſide FC and the angle AFC which is the angle BFC; alſo the two triangles ABG, BCG have a ſide and an angle common to both, viz. the ſide BG and the angle AGB which is the angle CGB.

I have now explained at ſome length the original and nature of the geometrical principles; and the inſtruments made uſe of for communicating this kind of knowledge, taking notice at the ſame time of ſome of the moſt unuſual forms of expreſſion and which are apt to perplex the learner at his firſt ſetting out.

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DISSERTATION II.

Concerning the nature and extent of geometrical demonſtration.

IN the former diſſertation, as much care as poſſible has been taken to avoid the ſuggeſting any circumſtance to the reader's imagination, which will not be neceſſary in the very firſt ſteps to be taken in this ſcience. I muſt therefore beg he will not conſider what has been ſaid as words of courſe, merely to ſcrape acquaintance, or any attempt to diſplay my own learning, but as ſeriouſly intended for his improvement. Nor would I have it thought that this proceeds from any want of inclination to be in the good graces of my reader, much leſs from want of capacity; for with very conſiderable deductions both of thought and trouble, like ſome other commentators I could have flouriſhed away, in an explanation of the eleventh common notion, upon the properties of the aſymptotes of the Hyperbola &c; if my intention had been to raiſe the admiration of the reader, rather than to fix his attention. Having therefore reſolved to ſacrifice every ornament to this ſingle conſideration; the learner, it is to be hoped, will endeavour to convince himſelf, or rather ſome more impartial judge, that he underſtands every thing mentioned in the laſt diſſertation before he proceeds to this; in which he will find the nature and end of geometrical demonſtration laid open with all the ſimplicity and perſpicuity which the commentator could give it; and with as much particularity as the patience of the reader, a faculty rather apt to be diſcompoſed by much exerciſe, could well be ſuppoſed to bear.

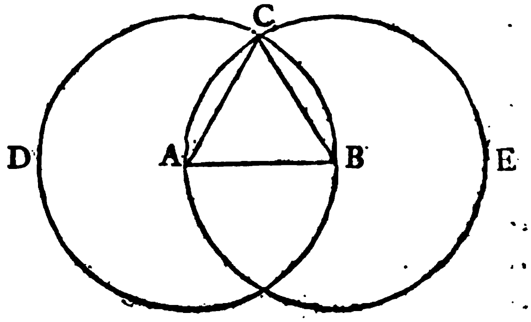
CHAP. I.

Containing an explanation of the two firſt propoſitions.

THE two firſt propoſitions are thus explained by Dr. Hooke "Euclid having premiſed his principles, he begins his method "of demonſtration, in which he takes no more for granted than "what he hath already laid down as eaſy and ſelf evident. His firſt "propoſition then is upon a right line given to make an equilateral "triangle. He hath defined in the fourth definition what he "means by a right line, namely that which lieth ſtraight between "the two extremes of it which are points; and what he means "by an equilateral triangle, namely, ſuch a one which hath all its "three ſides equal to one another."

"This firſt propoſition is a problem, which explains a way how "to do and perform the thing required, as well as ſhews how to "manifeſt the truth and certainty of the thing done: It contains "therefore and ſhews a double invention, without which or ſome "ſuch other thing, the propoſition can neither be done nor demonſtrated; "which inventions are called mediums or means by which "we attain to the end propounded or deſired. The end here "ſought is how from the ends of a line given to draw two other "lines, each equal to the given line, which ſhall meet in one and "the ſame point. It is certain that theſe lines muſt begin from "the ends of the firſt line given; but which of theſe to draw firſt, "and which way, and with what inclination to the former line, "that is with what angle, that is not yet known, and ſome invention "muſt be thought of, how to direct our ruler to draw it. "Well how ſhall this be done, ſince there may be infinite of "lines drawn from each of thoſe points, which ſhall every one of "them be equal to this line given? How then ſhall we among "thoſe infinite or indefinite number chuſe out the right? 'tis impoſſible "without ſome invention. Our author therefore helps "you to one, and one which you have already granted to be feaſable "in the third petition. Upon the center A, and diſtance AB, "draw a circle, ſays he, BCD; what then? To what purpoſe;

“ Why this circle then will give you a line, in which are contained “ all the points or ends of the infinite lines, which may be drawn “ from the point A any ways that ſhall be equal to AB. How ſo? “ Why by the fifteenth definition you are taught, that a circle is a “ plain figure bounded by or contained within one curve line, “ which is called the circumference; to which every right line “ drawn from a point in the middle, which is called the center, “ are equal to one another. But what are we yet the wiſer? How “ do we know which of theſe infinite lines we are to draw? To “ which of theſe infinite points that are in this circumſtance? To “ know this, you muſt do the ſame thing upon the point B: that “ is, upon the point B and diſtance BA, draw or deſcribe the circle “ ACE, which will give you all the poſſible infinite points in that “ plain; to the which from the point B right lines may be drawn “ equal to BA. Now then ſince theſe circles contain all the poſſible points of the lines equal to AB or BA that can be drawn “ from A or B. It follows that where thoſe circles interſect, there “ only muſt be the point to which thoſe lines may be drawn; “ namely at C and at the other point of interſection and no where “ elſe ſoever. Drawing therefore lines from A and from B to either “ of thoſe points as to C as AC, BC; you have done the thing “ that was propounded; namely upon the line AB given you have “ made an equilateral triangle ABC; which was deſired. This is “ the firſt part of the problem, and indeed the “ difficulteſt to find out; “ namely how to do the “ thing required; and in “ this part lie the greateſt “ difficulties of mathematical knowledge, to wit, “ in finding out the proper and true mediums “ or means to perform the problems required to be done, which “ for the moſt part are of the ſame nature with this, and conſiſt in the finding out the poſition of a point: for this problem might have been thus worded. A right line being given



“ as AB, to find a point as C, to which lines being drawn from “ the points A and B, they ſhall each of them be equal to one “ another, and to the line AB which is given: or two points A “ and B being given to find a third point as C, which ſhall have “ the ſame diſtance from A and from B that they have from one “ another. But our author not having given any definition of diſ- “ tance or equality, otherwiſe than may be collected from equality “ of the ſides of ſome figure; or of the rays or lines drawn from “ the center to the circumference of a circle, he chuſes rather to “ make uſe of an equilateral triangle to find out that propriety of “ a point ſo poſited.”

“ The ſecond part then of the propoſition is the demonſtrative “ part thereof, namely, to prove from the principles already laid “ down and granted and aſſented unto for true and certain, by a “ clear chain of reaſoning and deduction, that theſe lines AC and “ BC are each of them equal to AB and ſo equal to one another; “ and conſequently that the figure ABC bounded and limited by “ them is an equilateral triangle, according to the deſcription of “ that figure in the twenty fourth definition. The next thing then “ to be invented or found out is the medium or means of demon- “ ſtrating it to be ſuch; for this we have two. Firſt, the definition “ of the properties of the lines from the center of a circle to the “ circumference in the fifteenth definition, that they are all equal “ to one another. And ſecondly we have the firſt axiome, thoſe “ which are equal to one other are equal to one another. Firſt AB “ is equal to AC, becauſe they are right lines drawn from the center “ A to the circumference BCD; for by the fifteenth definition, as “ I ſaid, all ſuch lines muſt be equal. Next BA and BC muſt “ upon the ſame account be equal to one another, becauſe they “ alſo are lines drawn from the center B to the circumference “ ACE: Therefore both AC and BC are equal to AB; but by the “ firſt axiome thoſe which are equal to one other are equal to one “ another, therefore AC and BC are alſo equal to one another. “ Therefore the three ſides of the figure ABC; namely, AB, AC, “ and BC are equal to one another, and conſequently bound an “ equilateral triangle according to the twenty fourth definition,

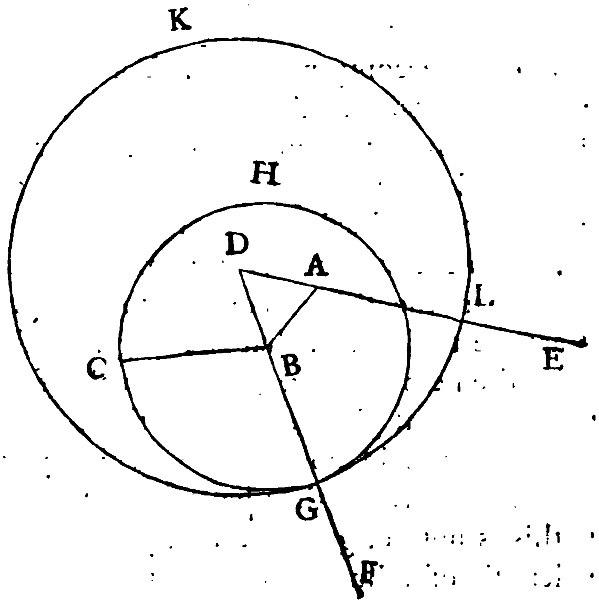
" therefore upon the line AB given, an equilateral triangle is made, " which was the thing to be done and proved."

" THE ſecond propoſition is alſo a problem.

" From a point given to draw a right line, which ſhall be equal " to another right line given. For the performing of this propo- " ſition, it being a problem, there is need of two mediums to be " invented or found out; the firſt is for the doing of the thing re- " quired; and the ſecond for the demonſtration of it; and both " theſe are to be fetcht out of the principles already agreed to, or " from the truth evidenced in the preceding propoſition: For we " are not to ſuppoſe any thing further known in this ſcience, and " therefore much leſs are we to make uſe of it. Searching there- " fore our ſtore, we have no other medium to make a line equal to " a line, than firſt by the help of a circle, defined definition 15: " which by the third *poſtulatum* is granted to be deſcribable upon " any center and at any diſtance: Or ſecondly, by an equilateral " triangle deſcribed by definition 24, which we learnt how to make " by the preceding problem: For as to the equal ſides of an iſoſ- " cles triangle, definition 25, or the equal ſides of a ſquare, a pa- " rallelogram or oblong ſquare, Rhombus or Rhomboeides deſcri- " bed definition the 30, 31, 32, and 33. Though their properties " are there defined, yet we are not taught how to make them as " yet, and conſequently can make no uſe of them, as *media* to " perform the thing required to be done by this problem. Nor " are we to ſuppoſe, that the length of the line given may be taken " by the help of a meaſure or a pair of compaſſes, and transferred " to the point given; becauſe thoſe are firſt not mentioned in the " principles laid down there, which you are to make uſe of, and " of no other, till they be accepted for principles undeniable: For " this is not yet granted, that you can with compaſſes, take a true " length of a line, much leſs that you can transfer it and ſet it in " another place. But you have granted that 'tis poſſible, upon a " point given at any diſtance, to deſcribe a circle, or ſuppoſe it ſo " done, which is ſufficient for the demonſtration, that being the " principal thing aimed at by our author, namely, to lay open to " the underſtanding the reſons and grounds of the properties of " quantities ſo and ſo qualified, that you may plainly ſee how and

for

“ for what cauſe things are thus or thus and cannot be otherwiſe.
“ For as to the moſt practicable and expedite ways of doing thoſe
“ things mechanically, and for other uſes, *that* belongs to another
“ part of mathematics; namely to the practical part thereof, which
“ is called practical geometry, or mechanical geometry, which
“ ought not to be learned till this be firſt known; but this which
“ our author treats of is ſpeculative geometry, and principally aims
“ at demonſtrations or explaining the proprieties of quantities to
“ the underſtanding. You ſaw clearly by the former propoſition
“ why ABC was an equilateral triangle; and there could be but
“ two ſuch made upon the line AB in the ſame plain; there being
“ but two points where in thoſe circles cut each other; thoſe circles
“ determining all the lines equal to AB that can be drawn
“ from the points A and B. His way then of performing this
“ problem is this: Let the right line given be BC and the point
“ given be A; from
“ which point a right
“ line is to be drawn
“ equal to the line BC.
“ Firſt draw a line from
“ B to A, which is
“ granted poſſible by
“ the firſt poſtulatum;
“ then by the former
“ propoſition upon this
“ line, BA make an equilateral
“ triangle BAD:
“ then upon the center
“ B and diſtance BC
“ deſcribe a circle, as
“ CGH by the third
“ poſtulatum; then by
“ the ſecond poſtulatum produce DB to F; then upon the center
“ D and diſtance DG draw the circle GKL, then as before, produce
“ the line DA to E; there ſhall AL be the line required to
“ be drawn from the point A equal to the given line BC.”



"This is the conſtruction of the problem, or the preparing of "the propoſition fit for demonſtration, by which you may clearly "underſtand the reaſons of it, deduced from the few principles "already laid down: For firſt, that BC is equal to BG is clear "from the fifteenth definition, which determines the propriety of "equality of the rays of a circle. Next that DBG is equal to "DAL, is as clear from the ſame definition, they being both rays "or lines drawn from the center D to the circumference GKL by "the conſtruction premiſed. Thirdly that DB is equal to DA, is "clear from the conſtruction; for DBA is an equilateral triangle, "two of whoſe ſides DB and DA are. Now by the third axiom "or common notion, if from equal quantities you take equal quan-"tities, the remainders ſhall be equal; if from DL you take DA; "and from DG, DB; the remainder AL ſhall be equal to the re-"mainder BG; but BC is alſo by the conſtruction equal to BG; "therefore ſince by the firſt axiom theſe two quantities which are "equal to one other quantity are equal to one another; therefore "BC and AL, being equal to BG, are equal alſo to one another; "therefore from the point A the line AL is drawn equal to the "line BC; which was the thing to be done and proved."

"Now though this way of demonſtration and reaſoning may "ſeem tedious and too long to detain the mind and attention in "the finding out the proprieties of quantities, yet 'twas the way "made uſe of by the ancients. And 'tis altogether neceſſary, eſpe-"cially in the beginning of this ſtudy, to accuſtom the mind to "attention and circumſpection, that it may receive nothing for "truth but what it ſees clearly by the reaſons and cauſes of it, "that thereby the mind may acquire an habit of intention, and "of examining the whole chain of conſequents from the firſt prin-"ciples to the truth evidenced. For the want of which, ſome "ſmall error perhaps may ſlip into the mind under the appearance "of truth, and thereby make all the ſubſequent reaſonings and "deductions unſound; and 'tis very much harder to clear and free "the mind from it when once received, than to prevent the recep-"tion thereof. There cannot therefore (in this ſtudy eſpecially, "not now to mention any other, where it is poſſible it may be "altogether as convenient, nay neceſſary) there cannot I ſay

" therefore be as I conceive to much circumſpection and caution " uſed in admitting principles, and furniſhing the mind with the " true grounds of knowledge; becauſe for the moſt part we are too " prone to take up every thing we hear upon truſt, without exami- " nation: we are too apt to run away with a thing, and think we " know it and ſee it clearly before we are ſure we do; and are " impatient of delay in examining and conſidering; whereas if the " mind be a little at firſt accuſtomed to this leiſurely and ſtrict " way of reaſoning, after it has got a habit it will make as much " diſpatch in receiving things with ſufficient examination, as ano- " ther ſhall without it. And the patience only is needful for the " moſt part at firſt to beget attention; nor is it peculiar to this " acquiſition alone; but we ſee it neceſſary, and practiſed in many " other things where a good habit is to be acquired; as in reading, " writing, muſic, drawing, and moſt other manual operations. " The roots and beginnings of knowledge, and practice too, are " bitter and tedious, but the fruits are ſweet and pleaſant; and " whoſoever attains the end, will never repent the time ſpent in " the beginning."

CHAP. II.

The ſame ſubject continued.

IN the preceding chapter I have given Dr. Hooke's explanation of the two firſt propoſitions, in order to ſhew the reader that I am not ſingular in my opinion, that Euclid is not to be underſtood, unleſs the learner beſtows that attention upon the firſt principles, which may enable him to carry along with him their full meaning and import. It is true that ſome confuſed facts concerning the properties of figures may be picked up by a very careleſs peruſal of this author: but whoever is ſatisfied with this, had better look for his knowledge ſome where elſe; becauſe he will meet with a great many impertinent interuptions to his ſcheme from the ſeveral ſteps of the demonſtration; and rather content himſelf with a kind of law evidence, by reſolving to conſent to every propoſition which is delivered as truth by two or more credible authors.

Although I approve in general of what Dr. Hooke has advanced, yet I am convinced that a more minute conſideration of the different parts of each propoſition will be neceſſary for the ſtudent who would wiſh to leave his prejudices behind him as he advances. I ſhall therefore point out theſe circumſtances, which he might be apt either to overlook or miſtake. Suppoſing the firſt propoſition of Euclid carefully examined, he will find it taken for granted that the two circles cut one another, and that this ſuppoſes the circles to be deſcribed upon the ſame even ſurface, or according to the geometrical language in the ſame plane, otherwiſe there could be no ſuch point as C. Again the ſuppoſing C to be a point implies that the two lines, the two circumferences of the circles have no breadth, otherwiſe the point would have parts. Likewiſe by ſuppoſing the triangle ABC to be an aſſignable or determinate magnitude implies that the points A, B, C have no magnitude for if they had parts various triangles differing both in ſides and angles might be deſcribed, and it would be impoſſible ever to arrive at a determinate concluſion.

But farther, the reader is to conſider that he has been reaſoning all this while, upon a particular ſtraight line of very inconſiderable length; He is therefore next to inquire, whether his conſtructions and concluſions are alſo particular. That they are not will appear from this, that no conſequence is ſuppoſed to follow from the ſtraight lines having any particular length, nor is any conſtruction undertaken upon that ſuppoſition; only that the line be finite that is, that we know its two ends.

But it happens rather unluckily, that though the ſcientific conſtruction be general we are nevertheleſs forced to take up with a particular one; which is very apt to create prejudices unleſs our attention be every now and then called to the general conſtruction; we muſt remember that, though we work with a ruler and compaſſes, the ſcience knows no ſuch inſtruments. Let us ſuppoſe the line AB ten miles in length; and that the ſame conſtruction is to be performed; we ſhall now get beyond the objects of our ſenſes; but if we have beſtowed the neceſſary attention to the problem, the underſtanding will have as clear and diſtinct a perception as it had before, when the line was perhaps not above three inches long.

Our author's plan obliges him ſtrictly to prohibit the uſe of any particular inſtruments; ſtraight lines and circles are ſuppoſed to be drawn and deſcribed by mechanical operations; and we are left to gueſs in what manner. It will however be prudent to acquire ſome one ready mechanic way at leaſt of conſtructing every problem. For I always ſuſpect any pretenſions to general truths which have not been collected from an examination of particulars. The ruler and compaſſes are ſuch exact minature repreſentations of the two poſtulates, that they are to be preferred to every other inſtrument for aſſiſting the imagination to follow Euclid's conſtructions: only the compaſſes are to be confined to their proper uſe, always to deſcribe circles, but never to meaſure diſtances. Indeed it is not always neceſſary, and often would introduce confuſion if the circles were compleatly deſcribed: it is therefore ſufficient to deſcribe as much of them as is neceſſary; as for inſtance, in the future application of this propoſition, it is obvious that it will be ſufficient to find the point C; becauſe the point where the circumferences of the circles cut one another gives us every thing neceſſary for the conſtruction.

We come now to the ſecond problem which is of a more complicated nature than the firſt; and yet there are very few, upon reading it, who do not fall in with the notion that Euclid has taken a great deal of unneceſſary pains to do what a ſingle opening of the compaſſes would have performed with equal or perhaps more exactneſs. But conſidering the ſcience *only* with a view to practice, the affairs of mankind are carried on upon a much larger ſcale than to be managed with a ruler and compaſſes; and we have only to ſuppoſe the line BC a mile in length, and this will anſwer the objection ſufficiently. The underſtanding is capable of reaching the general concluſion delivered in this propoſition; why then ſhould it be fettered with inſtruments? Euclid's contrivance is admirable and ſuited to the dignity of the human mind. According to his plan we are guided by inſtruments, but neither confined nor loaded with them.

After the ſtudent has fixed in his memory the conſtruction and demonſtration of this ſecond propoſition, it is more than probable that his concluſions will be limited to the particular figure which he be made uſe of and to that very poſition which the lines happened accidentally to take.

This is a very general prejudice, and will be removed by attending to the following directions. Let the learner conſider whether DA and DB produced will cut the circle CGH: and he will find that DB muſt cut it as it paſſes through its center, but that DA's cutting it depends upon the length of AB. Change the poſition of the point A and repeat the conſtruction and thus you will learn what lines have a fixt poſition or only an accidental one: deſcribe the equilateral triangle ABD upon the other ſide of the line AB and compleat the figure; alſo join the points A and C inſtead of the points A and B and repeat the ſame conſtructions; and by proceeding thus you will be able to conquer the prejudice of ſenſe and to acquire ſomething like a ſcientific view of the problem. For you will find no conſequence deduced from BC's having any particular length or the point A any particular poſition.

But the third propoſition will afford an opportunity of taking a ſomewhat different view of theſe two propoſitions. It is a problem. Two unequal ſtraight lines being given to cut of a part from the greater equal to the leſs. The ſimpleſt caſe of this propoſition would be, when both the lines are drawn from the ſame point: for making that point the center, and the ſhorter line the radius, the circle ſo deſcribed would ſolve the problem: but if the lines had any other poſition, a very different apparatus would be neceſſary; we muſt learn how to put a line at the extremity of the longer equal to the ſhorter line; and this cannot be done before we have learned how to deſcribe an equilateral triangle. I would therefore propoſe it as an exerciſe for the ſtudent, and his ſucceſs would be a proof that he was maſter of the ſubject, to ſet out upon a ſuppoſition that the book began with this third propoſition, which will now include the firſt and ſecond; and by this proceſs he will have their connexion with each other ſtrongly impreſſed upon his mind. His figure will conſiſt of five circles, and their uſe is to determine four points: ſuch a point as C in the firſt propoſition, G and L in the ſecond; and E in the third.

Whoever attempts to communicate knowledge to mankind, muſt write upon the ſuppoſition of a certain degree of improvement in thoſe to whom he addreſſes himſelf, otherwiſe his book can have no determinate end to anſwer. Euclid ſuppoſes his reader above the prejudice of ſenſe, and to have the ready uſe of his underſtanding, with a due command of atttention; and upon this ſuppoſition he has ſaid every thing neceſſary to convey the fulleſt information to his reader. But I write to a different ſort of people, to ſuch as are immerſed in the prejudices of ſenſe, and at the ſame time very thoughtleſs; to thoſe whoſe underſtandings are not diſpoſed to attend to the call of reaſon without frequent admonitions; and this is my apology for begging the ſtudent's attention to one thing more, before I finiſh this chapter; he will always find ſomething ſuppoſed or given in every propoſition, and perhaps nothing will contribute ſo much to a right underſtanding of the propoſition, as a diligent enquiry into the uſe made of the *data* or ſuppoſition, as for inſtance whether the reaſoning in the third propoſition, does not depend upon the ſuppoſition that AB is longer than C; and where the reaſoning would fail if that were not the caſe. Our author was very confident that his reader would know, that, without this part of the ſuppoſition, there never could have been ſuch a point as E: but I am affraid that mine truſted to his ſenſes for the real exiſtence of this very point; dont I ſee, ſays he, that the circle cuts it.

I ſhall conclude this chapter by reminding the reader again, that the uſe of the two circles in the firſt propoſition is to find the point C: and in the ſecond, that the equilateral triangle is deſcribed to fix the point D, which is to be the center of a future circle; whoſe radius DG is determined by the deſcription of the circle CGH; and laſtly that this circle itſelf is deſcribed to find the point L. And thus I leave the ſtudent to purſue his own meditations upon theſe three propoſitions, only advertiſing him, that, if he thought them eaſy upon the firſt reading, and ſtill perſiſts in the opinion that he then underſtood them, I am certain he knows nothing of the matter.

CHAP. III.

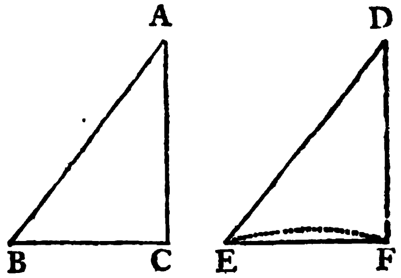
Concerning hypotheſes.

THE fourth propoſition is of a different kind from the three firſt; and is called a theorem. In the problems ſomething is required to be done: a problem therefore may be divided into two parts the conſtruction and demonſtration: for after ſhewing in what manner it may be done; it is neceſſary to prove that this is agreeable to thoſe principles which the nature of your ſcience obliges you to adhere to. Thus in the third propoſition it is totally inconſiſtent with the principles of this ſcience to take the line C in a pair of compaſſes; and by this means cut off AE from AB; for whatever inſtruments we uſe, it can be allowed to do nothing more by their aſſiſtance than to draw a ſtraight line and deſcribe a circle; the compaſſes then may be uſed in deſcribing a circle; but it can do nothing elſe without encreaſing the number of our mechanical problems, which Euclid has avoided with the greateſt care; and here it is to no purpoſe to alledge that it can perform the one as accurately as the other: but I deſire it may be obſerved that it is not admitted upon the ſcore of its accuracy, but as keeping in ſome reſpect to the idea of the poſtulate; for its merit in point of accuracy is realy nothing.

If geometrical knowledge could be communicated in the form of problems it would make a readier and ſtronger impreſſion upon the mind of the learner: for in the conſtruction of a problem the ſtudent is not barely a ſpectator but has an active part aſſigned him, to keep up his attention: but when a theorem is preſented to his conſideration, unleſs he has learned the uſe and management of an hypotheſis, he will be but an indolent and inattentive ſpectator; and to tell ſuch an one that the conſequence alluded to follows from the ſuppoſition is to addreſs him in a language which he does not underſtand. But for a more particular explanation I ſhall analyſe the fourth propoſition, which runs thus;

If two triangles have the two ſides equal to the two ſides, each to each; and have the angle equal to the angle, the *angle*, contained by the equal ſtraight lines: They will alſo have the baſe equal to the baſe; and the triangle will be equal to the triangle; and the remaining angles will be equal to the remaining angles, under which the equal ſides are extended, each to each.

This general enunciation of the propoſition will convey but a very indiſtinct perception of its meaning to a learner, eſpecially upon a firſt reading; this however may be improved and rendered diſtinct by the aſſiſtance of a figure, repreſenting two triangles in the ſuppoſed circumſtances; but here again a new difficulty ariſes, as this hardly ever fails to bring a prejudice along with it: for the triangles repreſenting not only the ſuppoſitions, but at the ſame time the inferences which are ſaid to follow from them; the ſtudent is at a loſs to diſtinguiſh what is given from that which is to be inferred from it; becauſe in all probability he will look upon the figure itſelf as the only ſource from which his knowledge is to be derived; and then he is as well convinced that the concluſions are true as the ſuppoſitions; and cannot conceive what it is he has to demonſtrate. And unleſs he be qualified to lay a proper ſtreſs upon the ſuppoſition, the demonſtration muſt appear to him an idle abuſe of words calculated only to perplex his underſtanding, eſpecially if he has already fixt his opinion by his compaſſes and other inſtruments. It is not eaſy to deviſe a ready remedy for this error; ſo that a man runs a riſque either of having no opinion concerning what is propoſed in the propoſition or an abſurd one; for of all the difficulties attending the acquiſition of the ſcience, this is the hardeſt to be got over. To reaſon accurately from a ſuppoſition is no eaſy matter; attention and habit will do a great deal, but above all a proper ſenſe of the difficulty of it. In the problems which we have already conſidered, the ſtudent has only to attend to the works of his own hands, in the firſt propoſition the ſtraight line AB is given him; but the two circles, and the two ſtraight lines AC and BC are his own manufacture; and likewiſe in the ſecond ſo is the whole figure except the point A, and the ſtraight line BC: this makes a diſtinction which nobody is ſo thoughtleſs as to overlook. But in a theorem like this fourth propoſition; where one has nothing to *do* but to *think*, the caſe is very different, eſpecially if the figure, which is intended to direct him, by miſtaking its uſe, ſhould become the occaſion of ſeveral prejudices. But to be more particular,



Let ABC and DEF be two triangles; having the two ſides AB and AC equal to the two ſides DE and DF; each to each, viz. the ſide AB equal to DE; and the ſide AC equal to DF; and the angle contained by BAC equal to the angle contained by EDF.

This now is the hypotheſis or ſuppoſition, which is to be conſidered as abſolutely certain; and a principle from which we are to reaſon as confidently, as from the firſt common notion that magnitudes, equal to the ſame, are equal to one another. Whatever might be collected from looking at the triangles, or by any other means, is to be entirely neglected; and not merely neglected, but even ſhunned as leading to the moſt pernicious errors. The ſtudent has only to examine the hypotheſis, one part after another, making uſe of the figure only to aſſiſt him in comprehending its meaning, which he might ſet about in ſome ſuch manner as this, repeating to himſelf, the two ſides I find are equal, but not any how; for they are ſaid to be equal each to each; the ſides of all triangles are inclined to one another; but here a particular inclination is ſpecified; they are ſaid to be equally inclined: this the attentive reader will comprehend perfectly, and be able to ſay I underſtand now what the author means; he affirms that all triangles which agree with one another in the circumſtances above mentioned cannot poſſibly fail in giving us the ſame conſequences. But let us ſee what conſequences he ſays will follow from his ſuppoſitions— They will alſo have *the third ſide or* baſe BC equal to the baſe EF; and the whole triangle ABC equal to the whole triangle DEF: and the remaining angles —*ſtop*, why that epithet remaining? Why, becauſe only one angle in each triangle being equal by the ſuppoſition, are there not then two in each triangle which remain to be conſidered? But to proceed, the remaining angles ſhall be equal to the remaining angles not both taken together, but each to each. EACH TO EACH, this though ſomewhat particular nevertheleſs admits of a latitude ſufficient to perplex an ordinary underſtanding; and without ſome additional circumſtance, muſt remain unintelligible even to the acuteſt. But the angles are particularly deſcribed and alſo fixed; becauſe they are to have the lines which are equal by the ſuppoſition extended under them; that is the angles under which AB and DE are extended are to be equal, as alſo thoſe under which AC and DF are extended; viz. the angle ACB or BCA equal to the angle DFE or EFD; and the angle ABC or CBA equal to DEF or FED. Now theſe are ſaid to be the conſequences which will certainly follow from the forementioned ſuppoſitions.

But it will be too much for the learner to attempt the whole demonſtration at once; let us therefore ſee what conſequence will follow from each part of the ſuppoſition taken ſingly. The principle by which they are to be examined is the eighth common notion, magnitudes which apply themſelves exactly to one another are equal. It is here ſaid that AB and DE are equal. What do you underſtand by this word *equal?* Certainly it muſt mean, that, if the ſtraight lines AB and DE be properly placed, they will apply themſelves exactly to one another. But what is the proper way of placing them? Put the point A upon the point D and the line AB upon DE; obſerve all this may be done whether the lines be equal or unequal; but then the point B will apply itſelf to the point E only when the lines are equal; for when AB is longer than DE the point B will be found beyond E; and when AB is ſhorter than DE the point B will be found between D and E. After a careful examination of theſe two particular lines; let us ſuppoſe each of them a thouſand miles in length; here our ſenſes forſake us, but ſurely our underſtanding unleſs it be weak indeed, will give us as poſitive a concluſion as in the other caſe; ſo that we may conclude univerſally that whatever be the length of AB provided only AB and DE be equal; and the point A upon the point D, and the line AB upon DE; the point B will always apply itſelf to the point E; which was the firſt conſequence to be examined. Alſo if the point A be upon D and the line AC upon DF; the point C will apply itſelf to the point F, as is obvious for the ſame reaſons; and that will always be the caſe whatever be the length of AC.

We come next to conſider in what circumſtances it is poſſible to place AB upon DE; and at the ſame time AC upon DF: Now this can always be done, if the angle BAC be equal to the angle EDF; becauſe when the point A is upon the point D: and AB upon DE; it follows from the equality of the angles that AC will take the direction of DF. But if the angles be unequal, although the point A be upon the point D; and AB upon DE; yet AC cannot poſſibly fall upon DF; becauſe AC will then take a direction, on the one ſide or the other of DF, according as the angle BAC is greater or leſs than EDF.

Laſtly let us ſuppoſe the two extremities of two ſtraight lines to be the ſame; may we not conclude from this that the lines are equal, and in the ſame direction, that is the one line will apply itſelf to the other; becauſe if it did not the two ſtraight lines would incloſe a ſpace. Therefore in this figure of ours, if it can be proved that the point B applies itſelf to E, at the ſame time that the point C applies itſelf to F; we may certainly conclude that BC will apply itſelf to EF and be equal to it.

Each of theſe ſuppoſitions ought to be examined frequently, and the conſequences which follow from them are to be ſtrongly impreſſed upon the memory; ſo that the very mention of the ſuppoſition may ſuggeſt the conſequence.

And here it may be proper to inform the ſtudent, that he is not to conſider this as any acquiſition of ſcientific knowledge; but only the conſequences which common good ſenſe cannot fail to draw, whenever theſe things become the ſubject of conſideration; theſe remarks are introduced not as teaching any thing new, but only to fix the attention. However the combination of theſe, as in the fourth propoſition, will carry us a ſtep beyond the common ſenſe of mankind: for he who firſt found out that the two ſides and included angle fix every part of the triangle beyond a poſſibility of change, had certainly more than a common notion of a triangle, and merited the high title of an *inventor*. The learner is now to endeavour to make himſelf maſter of the fourth propoſition, before he proceeds to the next chapter, in which he will find ſome remarks tending to correct, confirm or enlarge the notions which he may derive from it according to the different degrees of attention beſtowed upon it.

CHAP. IV.

The ſame ſubject continued.

THE reader is without doubt ſurprized to find this affair of *ſuppoſitions* repreſented as a difficulty almoſt unſurmountable; and not a little inquiſitive, we may ſuppoſe, after the reaſons of this ſtrange *phænomenon*. Our own indolence is the real cauſe, for ſimple reaſoning is hardly ſufficient to ſet the mind in motion without ſome external application. When a problem is propoſed, there is ſomething to be done, and, which is very much to our purpoſe, may be done wrong: this ſets us upon thinking; and by giving a preference to one method above another, we come to diſtinguiſh between what is right and wrong, and become ſo much intereſted as to give the queſtion a ſerious examination. But the caſe is very different in theorems, for the conſequences are abſolutely fixt by the ſuppoſitions, ſo that a mind without experience of ſuch ſubjects has nothing to engage its attention.

It would be abſurd to propoſe a problem of this kind: Suppoſe two triangles to have two ſides equal to two ſides, each to each; and the angles, contained by the equal ſides, equal; it is required to make the third ſide, equal to the third ſide; and the two remaining angles equal to the two remaining angles, each to each; namely thoſe under which the equal ſides are extended. Here there is no room for a conſtruction; becauſe according to the ſuppoſition, the ſides and angles cannot be otherwiſe than equal. However it is neceſſary to ſhew that theſe magnitudes are really equal; and the ſuppoſition alone is what we muſt truſt to, for bringing this about: and we are at a loſs becauſe, for this end, we begin to derive the conſequences from the ſuppoſition before we have examined the different circumſtances of which it conſiſts, or without underſtanding what isimplied in the ſuppoſition, when it reaches beyond the obvious meaning of the words in which it is expreſt; or when *more is meant than* directly *meets the ear*. The mind muſt have room to exerciſe itſelf in, as well as the body; and it is rendered almoſt inactive by being confined to a ſingle ſuppoſition. But if the ſtudent has patience this may be remedied by varying the ſuppoſition; and then you may have all the latitude and range of thought, which is allowed in the conſtruction of a problem.

Let us ſuppoſe then that only AB is equal to DE, and the angle BAC equal to the angle EDF; but that we are quite in the dark as to AC and DF.

When the point A is put upon D, and AB upon DE the point B will apply itſelf to E as before and AC will take the direction of DF; but it is impoſſible to determine any thing concerning the poſition of the point C, only that it will be found ſome where in the line DF or in DF produced.

Again let us ſuppoſe only the two ſides equal; then the point A being put upon D; and AB upon DE; the point B will as before apply itſelf to E, but we can make no uſe of the other part of the ſuppoſition; for we can determine nothing concerning the direction which AC may take, becauſe nothing is ſaid concerning the angles or inclinations of the lines.

But farther, let us ſuppoſe, that not only AB is equal to DE and AC to DF and the angle BAC equal to the angle EDF; but moreover, that AB is equal to AC and conſequently DE to DF: now by placing the triangles as in the propoſition, we ſhall find the angle ABC equal to DEF; and not only that, but, by putting A upon D, and AB upon DF the point B will now apply itſelf to F becauſe AB and DF are equal by the ſuppoſition; and alſo the angle ABC will be equal to the angle DFE: but ABC has been already proved equal to DEF; therefore by the firſt common notion the angle DEF is equal to DFE: that is the angles at the baſe of the triangle DEF are equal: But this concluſion does not depend entirely upon DE's being equal to DF; but it is required beſides that BA ſhould be equal to AC and to each of the lines DE, DF and likewiſe the angle BAC equal to EDF. So that this caſe can by no means be conſidered as including the fifth propoſition, in which there is no other ſuppoſition but the equality of the two ſides; and as we muſt have an angle equal to the angle at the vertex of the iſoſceles triangle, this angle muſt be made, which, in our preſent circumſtances, will not be found ſuch an eaſy matter. And ſo much for hypotheſes or ſuppoſitions.

CHAP. V.

Containing a critical examination of the fourth propoſition.

OBJECTIONS are brought againſt this propoſition, as if the demonſtration proceeded by a mechanical application of the two triangles to one another. But whoever ſtarts ſuch an objection as this, has not the moſt diſtant conception of the demonſtration; or elſe he muſt be ignorant of the nature of a mechanical application. And a particular examination will put this beyond all doubt.

Suppoſe two triangles of braſs, made as accurately as poſſible, all their different parts correſponding exactly. When we apply them to one another, all that we can ſay is, that as far as our ſenſes can judge, the parts ſeem to agree. Now what knowledge is got from this? Certainly by this application, no property of a triangle can be diſcovered; we may form a concluſion concerning the accuracy and neatneſs of the workmanſhip; but nothing farther. It is impoſſible to ſay from this that any thing is equal, but what is made equal; no doubt a very curious diſcovery and tending greatly to the enlarging the boundaries of the ſcience!

The miſtake here ariſes from not conſidering that it is impoſſible to make the parts of the figure which are to repreſent the ſuppoſition unleſs you make at the ſame time the parts expreſſing the conſequences which, it is ſaid, will follow. The parts repreſenting the ſuppoſition are to be tried and examined by the eye, as well as thoſe which repreſent the conſequences; ſo that if any conſequence follow it muſt follow from nothing. And the moſt attentive ſtudent will retire from this contemplation, with a very curious piece of information; namely, that he had ſeen two braſs figures ſo contrived as to fit each other exactly.

We now proceed to conſider the application made uſe of by Euclid in this fourth propoſition. And firſt it is to be obſerved that his triangles are required to have no particular poſition; therefore the point A may be ſuppoſed to be upon D as well as any where elſe; and likewiſe the line AB upon DE; and all this might happen to any two triangles not determined to any particular poſition. Then AB being to DE the point B muſt apply itſelf to E, it is impoſſible to conceive it to have another ſituation, which is very different from ſaying that as far as we can judge by our ſenſes it ſeems to be ſo; again if the angle BAC be equal to EDF, the line AC muſt take the direction of DF; and it is impoſſible to conceive it to take any other; and ſo likewiſe the point C muſt apply itſelf to F, whenever the line AC is equal to DF: and ſo on to the coincidences of the different parts of the triangles; and all this is as different from a mechanical application, as light is from darkneſs. For ſuppoſing what is taken for granted, it is impoſſible to conceive the conſequences not to follow; and this is certainly ſcience if there be any ſuch thing in the world.

But again it is alledged, why may not this method of applying be extended ſtill farther by making it an inſtrument for conſtructing problems? as for inſtance in the ſecond propoſition; why may we not ſuppoſe the line BC ſo placed that B may be upon A, and the thing required is done? For a very good reaſon, becauſe BC and the point A have each of them a fixt poſition already.

Upon the whole then we may conclude, that this method of application is perfectly ſcientifical; and that whenever two triangles agree in the circumſtances mentioned in this ſuppoſition; the conſequences will always neceſſarily follow; the baſes will be equal and the remaining angles, each to each, viz. thoſe under which the equal ſides are extended: Becauſe no conſequence is deduced from the lines having any particular length, but only from their being equal: nor is it ſuppoſed that the angle contained by the ſides is any particular angle, but only that the angles are equal. And this may ſuffice for an anſwer to the objections commonly brought againſt the demonſtration of this propoſition.

CHAP. VI.

Containing an explanation of the fifth propoſition.

ONCE upon a time a certain father reſolving not to be impoſed upon by reports, determined to examine into his ſon's progreſs in this ſcience, produced the book, and required him to demonſtrate a propoſition to which he referred: the young man though unacquainted with the ſubject, taking courage from his father's ignorance, began very impudently in ſome ſuch manner as follows; Becauſe the angle ABC is equal to the angle CBA, therefore the angle DEF is equal to the angle CEF &c &c; ringing the changes upon ſides and angles, until he had ſpun out his demonſtration to a decent length: and then kept ſilence in expectation of his fathers's opinion; who with a grave and important countenance remarked, "This is what we call demonſtration."

Every one is ſenſible, that it is contrary to common ſenſe to imagine that the letters of the alphabet thus repeated can have any meaning; but the indolent reader ought to be informed, that the repeating ſuch phraſes, in a regular order, mends the matter but very little, unleſs they convey to the mind their proper meaning: and unleſs the fourth propoſition be well underſtood, the moſt of that which follows will be nothing but an inſignificant jargon. As the whole ſcience therefore depends ſo much upon an accurate and comprehenſive view of this propoſition, it would be proper for the learner, before he proceeds farther to take the opinion of ſome acquaintance ſkilled in theſe matters; who, by a particular examination might be able to determine, how far he can be properly ſaid to underſtand it; and this friend is authorized, upon his failure in any point, to admoniſh him, by ſaying, "this is "what we call demonſtration."

I myſelf can trace every miſtake concerning the following propoſitions, or partial conception of their meaning, up to my ignorance of the full import and meaning of this propoſition. For it is by no means to be underſtood as applicable only to ſuch triangles, as one may make uſe of to aſſiſt the imagination in tracing the ſteps of the demonſtration; but as carrying with it this extenſive and general meaning, that all triangles which have two ſides equal to two ſides, each to each, and the angles, contained by thoſe ſides, equal; that all ſuch triangles, I ſay, have their baſes equal; and the two remaining angles in every triangle, equal, each to each; viz. thoſe angles under which the equal ſides are extended. Or more properly that all ſuch triangles, whatever be their number, are but one and the ſame triangle. Or otherwiſe that the two ſides, and the angle contained between them, fixes every part of the triangle, beyond a poſſibility of change. But I ſhall now proceed to conſider the fifth propoſition. And the reader may recollect an attempt in the end of the fourth chapter, to derive this propoſition from the fourth, or rather to make it only a particular caſe of that propoſition, when the two ſides of each triangle are equal to one another; and I there gave the reaſons why it could not be conſidered as ſuch: nevertheleſs the artifice of this propoſition will be the better underſtood by proſecuting that ſcheme a little farther, producing the equal ſides, and proving the angles below the baſes of ſuch triangles equal, in the ſame manner as the equality of thoſe above the baſe was there demonſtrated.

It was then obſerved that the ſuppoſition was too complicated for this purpoſe; becauſe not only DE and DF were to be equal to each other and to AB and AC, for this might have been allowed; but alſo the angles between the ſides were to be equal, which could by no means be allowed; becauſe according to the fifth, only one angle being given, the other was to be made equal to it; which is not at preſent poſſible.

If the reader has proſecuted this ſpeculation far enough, he will certainly admire the ingenuity of our author for his contrivance to make theſe angles equal, in a very elegant manner indeed: and which, it is curious to obſerve, neceſſarily requires that the ſides of his two triangles ſhould be unequal, For he makes this very angle which we are conſidering common to both, by making the ſides of the triangles take the ſame direction, the ſhorter ſide of the one being upon the longer ſide of the other.

But now it will be proper to turn to the demonſtration itſelf; and after a careful examination of its different parts, I would recommend a particular attention to the uſe which might be made of it, in order to impreſs the laſt propoſition more ſtrongly upon the memory. The great advantage of problems above theorems, for fixing the attention of the learner, has been mentioned already: and here a little of that advantage may be gained, by the particular view of the fourth exhibited in this propoſition; which will teach us how to repreſent by an actual conſtruction, the ſuppoſitions in the laſt propoſition: becauſe, if we draw two undetermined ſtraight lines, making any angle; we can cut off AB equal to AC; and AF to AG; and, by joining BG and FC form two triangles, with all the parts of the ſuppoſition in the laſt, accurately repreſented; and this not in imagination, but conſtructed by ourſelves; and the conſequences, not made, as was objected to the mechanical triangles, but left to follow from the conſtruction; the accuracy of which, the ſlow of imagination may examine by inſtruments; and thus get a kind of palpable evidence for the truth of the propoſition; and by varying the inclination of the lines, and the length of the ſides, climb up by degrees to ſomething like a ſcientific conception of its meaning.

The other two triangles FBC and CGB, though they repreſent a particular inſtance of the fourth propoſition, are not ſo fit for this purpoſe, not being ſo general as the firſt two triangles nor ſo much in our power, or rather indeed they are not at all in our power: We may vary the angle and the two ſides in the triangle ABG at pleaſure; but we have no power over the ſides, or angle BFC of the triangle FBC; for they become ſuch as may happen from the conſtruction of the firſt triangle: therefore the epithet *any* could not be ſo properly applied to the triangle FBC as to ABG.

I have recommended to thoſe whoſe imaginations are ſlow or inactive, theſe conſtructions and inſtrumental proofs of the ſeveral concluſions which follow from the ſuppoſitions in the fourth propoſition. And I now, rather more earneſtly, recommend the ſame to thoſe whoſe imaginations are diſpoſed to out run their judgement, leſt they ſhould ſnatch the concluſions without the premiſes.

I have only one remark more to make upon this propoſition, which the judicious reader has already made for himſelf, namely, how little there is to attend to, in this very formidable propoſition, this *pons aſinorum,* upon a ſuppoſition that the fourth is perfectly underſtood: for except the circumſtances of taking the equal angles CBG, BCF from the equal angles ABG, ACF; it has nothing peculiar to itſelf; every thing elſe conſiſting of particular proſpects of the fourth propoſition.

But if any one is deſirous to have ſome kind of reaſon for fixing his attention a little longer upon this propoſition, by taking a ſomewhat different view of the ſubject; let him ſuppoſe, inſtead of this fifth propoſition, it is required to demonſtrate that all the angles of an equilateral triangle are equal; for inſtance ſuppoſe the angles of the equilateral triangle ACB deſcribed in the firſt propoſition. Produce CA, and CB; and the ſame conſtruction and demonſtration made uſe of in this fifth, will prove that the angle CAB is equal to CBA; and by producing AB and AC; in the ſame manner it may be proved that the angle ABC is equal to ACB; but it has been already proved that ABC is equal to CAB therefore, by the firſt common notion, the angle CAB is equal to ACB &c.

I ſhall conclude this chapter by deſiring the reader to take every opportunity of correcting a prejudice, which it will require all his art to remove. We cannot help drawing conſequences from the very poſition which the lines accidentally happen to take in that particular figure which we reaſon upon, though this particular poſition make no part of the ſuppoſition. The ſtudent may convince himſelf of this, if he read the fourth propoſition by the aſſiſtance of the figure which belongs to it; and then again making uſe of the two triangles ABG and ACF in the figure to propoſition fifth; making the ſame ſuppoſition in both caſes. If the conſequences are ſuggeſted to his mind more readily by one ſet of figures than another; this can ariſe from nothing but the ſtreſs laid upon the magnitude and ſituation of the triangles, which, by the very ſuppoſition have no particular magnitude or ſituation. But this prejudice will be leſſened, and gradually removed by varying the poſition of the figures, turning them upſide down; and by ſuch other methods as will readily occur to the attentive reader; but above all by enlarging the figures, ſetting the imagination to work until they ceaſe to be the objects of our ſenſes. The center of the moon, when in the equinoctial, joined by ſtraight lines to the two poles of the Earth, would make a very proper iſoſceles triangle with which the ſtudent might finiſh his ſpeculations upon this propoſition.

CHAP. VII.

Concerning indirect demonſtrations.

INDIRECT demonſtrations are generally uſed, when the propoſition is the converſe of ſome other which has been already demonſtrated. A propoſition is ſaid to be the converſe of another, when the hypotheſis or ſuppoſition in the one is the thing to be demonſtrated in the other, and the contrary; thus the converſe of the fourth would be; ſuppoſing the baſe BC equal to the baſe EF; and the angle ABC equal to the angle DFE; as alſo, the angle ACB equal to the angle DFE; that then the two ſides BA and AC will be equal to the two ſides ED and DF, each to each; viz. thoſe ſides which are extended under the equal angles, that is AB equal to DE and AC to DF; as alſo the angle BAC contained by the two ſides equal to EDF.

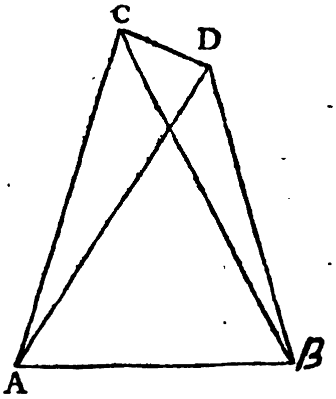
Now this propoſition admits of a direct demonſtration; for if the point B be put upon E and BC upon EF, the point C will apply itſelf to F, becauſe BC is equal to EF; and BC applying itſelf to EF; alſo BA will apply itſelf to ED, becauſe the angle ABC is equal to DEF. Certainly for the ſame reaſon CA will apply itſelf to FD, that is, becauſe the angle ACB is equal to DFE: the point A will therefore apply itſelf to D; and ſo AB will be equal to DE and AC to DF and the angle BAC will be equal to EDF. &c.

But if we ſuppoſe the angles at B and C, not only equal to thoſe at E and F, but alſo equal to one another; we may prove AB to be equal to DE as before; and then by putting B upon F and BC upon FE we can prove that C will apply itſelf to E; and that the line AB will be equal to DF: but it has been demonſtrated that AB is equal to DE, therefore by the firſt common notion DE is equal to DF; and no doubt ſome authors would produce this as a demona demonſtration of the ſixth propoſition; but it is liable to the ſame objections, as it converſe, which have been mentioned already.

But inſtead of going any ſuch way as this to work Euclid has given a very elegant demonſtration of the ſixth propoſition, which is the converſe of the fifth; which the reader may pleaſe to turn to and examine; after which he may proceed with the following remarks.

And firſt it will be worth his while to obſerve the ingenuity which Euclid has diſplayed, in making the angles which are equal by the ſuppoſition, the angles between the equal ſides of his two triangles; and BC being a ſide common to both it remains only for him to make BD equal to AC; and to join DC. And here it may be aſked, and it is alſo the moſt important queſtion which can be put with regard to this propoſition, how comes it to paſs that he cannot find by his conſtruction that the points D and A are the ſame? It is true ſuch an inſtrument as the compaſſes would find out this if it diſcovered any thing; but this uſe of the inſtrument is rejected for reaſons given already. For the ſolution of this difficulty therefore the ſtudent may pleaſe to turn to the third propoſition, and conſider the apparatus made uſe of, for cutting off, a part from a line equal to another: and he will find that the ſuppoſing the problem poſſible implies that the laſt deſcribed circle muſt cut AB that is that the point D muſt be between A and B. And when we find that this ſuppoſition leads to an abſurdity what are we then to conclude, but that the problem is really impoſſible; which never can be if AB is greater than AC; therefore it is not greater. And I appeal to any one, whether the demonſtration is either difficult, obſcure or inconcluſive when it is conſidered in this manner. Nothing can be ſo ſenſeleſs as the objections uſually made to indirect demonſtrations. Every demonſtration may be loaded with objections until it becomes indirect: the reader will find a ſpecimen of that kind in my remarks upon the firſt propoſition; where a caviller is introduced denying that the equilateral triangle there deſcribed is a fixt magnitude. But our author never carries his reaſoning farther than good ſenſe requires; ſo that it is only ſuch propoſitions as this ſixth, which take an indirect form in his hands. And this kind of reaſoning will not apply unleſs we know how many ways the thing may be; as in the preſent inſtance AB muſt be equal to AC or longer or ſhorter: or elſe we muſt get at ſuch direct conſequences from one part of the ſuppoſition, as will overturn the other: as in the next propoſition, which I muſt now beg the reader to peruſe: after which I ſhall be ready to lend him my aſſiſtance in removing ſuch difficulties as uſually lie in the way of beginners.

It is ſuppoſed here, that AC is equal to AD; and at the ſame time BC equal to BD. The learner will do well to fix his attention entirely upon this: firſt tracing the conſequences which follow from the ſuppoſition of AC's being equal to AD as far as they will go; which he may eaſily do; as it is only that the angle ACD is equal to ADC.



He is next to obſerve the conſequences which follow from the poſition of the lines; and theſe conſequences are, that the angle ACD is greater than BCD, which follows from this common notion that the whole is greater than its part; and alſo that the angle BDC is greater than the angle ADC, which follows from the ſame principle: and with a diſtinct impreſſion of theſe things upon his mind, it is impoſſible to miſs the concluſion that the angle BCD is much leſs than BDC: and this point being once gained; he is next to turn his thoughts to the other part of the ſuppoſition; in which it is pretended that BC is, alſo at the ſame time, equal to BD: and the conſequence which follows from this part of the ſuppoſition cannot fail to engage his attention; being no other than this, the angle BCD is therefore, by the fifth propoſition, equal to BDC: When we ſuppoſe AC equal to AD we muſt conclude that the angle BCD is much leſs than BDC; but theſe very angles muſt be equal upon the ſuppoſition that BC is equal to BD; is not this ſaying in the ſtrongeſt terms that the two ſuppoſitions are inconſiſtent with one another; that is, that it is impoſſible for AC to be equal to AD; and at the ſame time, BC to be equal to BD. Which was to be demonſtrated.

It will be a very neceſſary and uſeful taſk for the reader to ſet himſelf after he has finiſhed every demonſtration; to examine whether ſome conſequence has been drawn from every part of the ſuppoſition; for though it may not always be convenient for the author to draw them, yet the reader ſhould always do it for himſelf; for if the ſuppoſitions have no conſequences to the purpoſe either expreſſed or underſtood; they ought by all means to be omitted; and if the author truſted to the reader's ingenuity for *finding them out* he neglects his duty if this part of his buſineſs be overlooked. The neceſſity of this practice being thus made evident; I ſhall explain my meaning by a particular example; it is ſaid that AC and AD are to have the ſame extremity; but no uſe is made of this expreſly in the demonſtration: and yet without this the demonſtration could not proceed; becauſe ACD could not be a triangle unleſs theſe lines had the ſame extremity: neither could it be a triangle unleſs C and D were different points. It is moreover ſaid that they are to be towards the ſame parts: now if the points C and D were on different ſides of the line AB; the angles ACD and BCD would either be two diſtinct angles and the one not a part of the other; or if BCD be a part of ACD, then ADC would not be a part of BDC; which is abſolutely neceſſary for bringing out the concluſion which we aim at. Laſtly it is ſaid that the equal lines are to be terminated in the ſame extremity; and without this neither ACD nor BCD would be iſoſceles triangles; and then no inconſiſtency could follow, reaſon as long as we pleaſed. All this will be obvious by taking the points C and D on different ſides of AB, and joining CA, CB; DA, DB and CD; ſo that CD may cut AB or fall beyond the point B. And indeed it is very difficult to underſtand any general propoſition, without ſome repreſentation of all the different poſitions which the lines can take: and the reader cannot finiſh this propoſition better than by ſuppoſing the point D within the triangle ACB, and to aſſiſt his imagination, by making AC and AD equal with a pair of compaſſes; his demonſtration will reſt upon the ſame principles; only producing AC and AD; he is to reaſon upon the angles below the baſe; but every thing elſe will be the ſame as in Euclid's demonſtration.

I ſhall conclude this chapter with my opinion of indirect demonſtrations, which ſome very ignorantly object againſt. It would be abſurd to ſpeak of the certainty of a demonſtration: as to their uſe therefore, conſidered as an exerciſe to the mind, I think they are preferable, for ſeveral reaſons to direct demonſtrations. Becauſe in a direct demonſtration, there is generally ſome circumſtance which catches the ſenſes; and not being too difficult of perſuaſion, where our own eaſe is ſo much concerned, we are diſpoſed to reſt very well ſatisfied with that, though it may convey, but a very imperfect notion of the full extent of the demonſtration. But the contrary happens in ſuch demonſtrations as are indirect; for the reaſoning being generally at variance with the ſenſible repreſentations of the magnitudes, it requires ſome effort of the mind to get beyond the prejudice of ſenſe, ſo as to comprehend the force of the reaſoning: and when we cannot conquer our indolence or command our attention, it is a very decent excuſe to lay the blame upon the nature of the demonſtration. But I have been ſo particular in my examination of theſe two propoſitions, that the attentive reader can hardly want any farther information upon this head.

CHAP. VIII.

Of geometrical demonſtration.

IT will now be proper for the reader to endeavour, from a careful examination of theſe firſt ſeven propoſitions, to collect as accurate a notion as poſſible of a demonſtration, according to the rules which Euclid has preſcribed to himſelf.

We are apt to ſatisfy ourſelves of the truth of things, in the eaſieſt and ſhorteſt way we can; and even when we chuſe to conſider the properties of magnitude, inſtead of having recourſe to ſome general principles, we truſt to mechanical inſtruments; and very often to vague conceptions collected at random from accidental obſervation. But our author's plan is very different from this: The definitions, poſtulates and common notions, are the only foundation upon which his geometrical reaſonings are founded;

and when he mentions any thing as a property of a figure, he conſiders it as incumbent upon him to ſhew that it follows from or agrees with the definition of that figure; and this he does by the aſſiſtance of ſome common notion, poſtulate or other definition. And whatever property is not reducible in this manner, he conſiders as impoſſible to be demonſtrated; though it might be a very obvious truth from other principles. Truth therefore is not ſo much his object as conſiſtency; it does not appear directly to be any part of his buſineſs to collect as many uſeful facts as poſſible concerning the properties of figures; but only to be convinced that what he does produce have a ſolid foundation, making every part acquire additional ſtrength by the conſiſtency of the whole.

He will never allow it to be ſaid that ſuch a property is ſo ſimple as not to require a demonſtration; for to alledge this as a reaſon for taking any thing for granted, would be, upon his plan the greateſt abſurdity. For it may be ſaid, if it be ſo very plain and obvious, whence does it derive this obviouſneſs? If it be the property of a figure; that figure is defined to be ſo and ſo. How does it appear that it is conſiſtent with that definition? If you point out the conſiſtency, you have demonſtrated the propoſition; but if that cannot be done, it may be true according to your principles, but mine lead me abſolutely to reject it.

That this is Euclid's plan will appear from the whole of his work, even to the aſtoniſhment of the attentive and judicious reader; when it is conſidered how his invention muſt have been perplexed, by the obvious, but inaccurate experimental teſts, which would for ever be preſenting themſelves to his imagination; and how his patience would be tired out by the number of ſteps, which he forſaw to be neceſſary, before he could arrive regularly, at ſome concluſions ſeemingly very ſimple indeed. A man would feel himſelf in a very ackward ſituation, who ſhould ſet about meaſuring magnitudes, without being able to judge when one ſtraight line was longer than another; and yet it is wonderful, by how many ſteps, this very teſt is removed from the firſt principles; and which are all neceſſary for giving it a ſcientific ſtability. We find this delivered in the nineteenth propoſition of the firſt book; but as this is rather foreign to my preſent purpoſe, and beyond the ſuppoſed knowledge of the reader, it will be ſufficient to have mentioned it by the bye, only as a fact to illuſtrate my meaning. I ſhall therefore now beg the attention of the ſtudent, while I review the firſt ſeven propoſitions, in ſupport of my opinion concerning Euclid's plan of demonſtation.

In the firſt propoſition his principles are the third poſtulate; the firſt poſtulate; the fifteenth definition; the firſt common notion; and laſtly the twenty fourth definition. He is not ſatisfied to make an equilateral triangle any how; but he makes it by the aſſiſtance of his own inſtruments and principles. The ſecond propoſition is conſtructed and demonſtrated by the firſt poſtulate, the firſt propoſition, the third poſtulate, the fifteenth definition and the third common notion; and we have alſo here a proof how tenacious our author is of his principles; for it is not abſolutely neceſſary that in this ſecond propoſition, an equilateral triangle ſhould be deſcribed upon AB; becauſe an iſoſceles triangle would have anſwered the purpoſe; for no conſequence is drawn from AB's being equal to AD; as it would be ſufficient if AD and DB were equal; but he could not make an iſoſceles triangle, conſiſtent with his own plan and principles. The third propoſition reſts upon the ſame foundation; its demonſtration being ſupported by the ſecond propoſition, and the third poſtulate.

In the fourth propoſition the principles which we reaſon from, beſides the ſuppoſition, are the eighth and twelfth common notions; to which every part of the demonſtration may be reduced; for the application of the triangles to each other muſt be allowed to be a proper principle to reaſon from, otherwiſe it is difficult to conceive what uſe can be made of this eighth common notion, which is the definition of equality; it will not matter much whether this putting of the triangles be reckoned a conſtruction or not; I am rather inclinable to conſider it as a thing of its own kind, and not to be underſtood either as an hypotheſis or conſtruction: I know Euclid makes uſe of the expreſſion the point being put, which ſeems to imply a kind of conſtruction; and yet it would be doing no great violence to the language, to paraphraſe it in this manner; the triangles have no particular poſition; that is they have no relation to any points, lines or angles, which can be affected by our giving them a particular poſition; let us ſuppoſe them then to have ſuch a poſition, as it is neceſſary magnitudes ſhould have before that eighth common notion can be of any uſe; that is let us ſuppoſe their ſituation ſuch, that the one is applied to the other; and the point A put upon D &c.

It is plain that Euclid himſelf found it impracticable to give this principle a more ſcientific form; and therefore inſtead of making a definition of it, left it among the common notions; but whatever opinion we adopt; it will anſwer our purpoſe, becauſe every thing is inferred from the principles, which are laid down as ſuch. If we were required to place triangles upon each other, which had a determinate poſition before, I would certainly conſider it as a conſtruction; but in the preſent caſe, I conſider it only as a neceſſary apparatus before any conſequence can be drawn from the eighth common notion. And as to the objections, upon a conceit that this is a mechanical application, they have been very fully anſwered already; ſo that upon the whole there is nothing in this propoſition, but what agrees with the notion of a demonſtration which has been delivered in the beginning of this chapter.

The fifth propoſition ſeems by the references to depend upon the four firſt propoſitions and the third common notion; but as the conſtruction required is only the ſimpleſt caſe of the third, the firſt and ſecond propoſitions are not neceſſary for cutting off AG equal to AF. But the ſixth requires the four firſt propoſitions; becauſe the ſimpleſt caſe of the third, will not be ſufficient for cutting off BD equal to AC; and, what I muſt beg the reader particularly to attend to, neither can it be done by a ruler and compaſſes; becauſe there is ſomething ſingular in this conſtruction: for it is here required to do, what we afterwards find impoſſible to be done; and the unlucky circumſtance is, that whoever truſts to a ruler and compaſſes will come to the knowledge of this too ſoon; a pair of compaſſes lets him into the ſecret immediately, before his mind is prepared for it by any ſcientific information; and hence will ariſe a groſs miſconception of the author's meaning; becauſe the reaſoning muſt appear uſeleſs and unneceſſary, being intended to diſcover to a man what he has found out already. In all ſuch caſes as this therefore our ruler and compaſſes are to be laid aſide, and ſome method deviſed which may be more conſiſtent with the nature of the poſtulates as here applied : for by keeping ſtrictly to the information wchih we get from them, we never can diſcover that we have been attempting to perform what is impoſſible, before we are led to the abſurdity of concluding the triangle ABC to be equal to DCB. I would recommend the performance of ſuch conſtructions as this, by the hand unguided by any inſtrument ; as this will favour the condition of the propoſition, becauſe ſuch a conſtruction can make no diſcovery before the proper time. Not that I would *ſuppoſe* it performed ; as that might foſter prejudices, but rather repreſent every ſtep of the conſtruction ; BC being joined already I would deſcribc upon it an equilateral triangle producing the ſides &c. as directed in the remarks on the third propoſition, until the laſt deſcribed circle cut AB in ſome ſuch point as D ; becauſe the whole conſtruction will be neceſſary to convince the reader that he has no ſcientific notice of the impoſſibility of the problem from the conſtruction, but muſt wait for it until he come to the abſurdity. And hence it appears that Euclid requires every part of this propoſition to be referred to the firſt principles ; contrary to the ſentiments of thoſe ſuperficial readers, who imagine that they have only to ſuppoſe a point D, and the thing is done.

The demonſtration of the ſeventh is reducible to the fifth ; and the ninth common notion. And thus it appears, that Euclid's intention is not to ſhew that the propoſition is true in any manner ; but that it is immediately connected with, and depends upon his principles, or in other words that his aim is not to perſuade but to demonſtrate.

CHAP. IX.

In which is explained the geometrical meaning of the words finite and infinite.

“ THE human underſtanding, *ſays Bacon,* ſhoots itſelf out, “ and cannot reſt ; but ſtill goes on though to no purpoſe. Thus “ 'tis inconceivable there ſhould be any bounds to the univerſe ; “ yet it conſtantly, and, as it were, neceſſarily recurs, that there

" muſt be ſomething farther. So again it cannot be conceived how " eternity ſhould have flowed to the preſent time: and there is the " like ſubtilty as to the infinite diviſibility of lines, &c. all ariſing " from the weakneſs of human thought." And whoever chuſes to indulge this *weakneſs of thought* will be put into a fine train by reading what *Locke* has advanced upon the idea of infinity. But it is the end of all mathematical learning, inſtead of lending any aſſiſtance to encourage this kind of dreaming, to root out ſuch a diſpoſition from the human mind; and inſtead of ſuch fooleries, to give it ſomething to employ itſelf upon ſuitable to the powers and faculties of a human ſoul. An underſtanding which has any force will always ſhake off ſuch dreams; the indulging in which is the ſign of a weak intellect. For the philoſopher and the idiot agree perfectly in their notions of infinity; and the difference between them is only diſcovered when they come to a compariſon of determinate things and quantities.

The geometrical notion of infinity, as far as it regards extenſion, is derived from the ſecond poſtulate; let it be taken for granted that a ſtraight line may be continued directly forward: The geometricians never trouble themſelves with multiplying any aſſignable parts of extenſion, in order to be convinced that after they have advanced millions of miles, they are ſtill as far from the end of an infinite line as they were at their firſt ſetting out.

The two ſtraight lines DE, DF in the ſecond propoſition, are infinite in the full and proper geometrical ſenſe; and ſo likewiſe are AD and AE in the fifth; and the third propoſition may be applied to conſtruct this, becauſe AF is leſs than AE: but why is it leſs? Becauſe AF is a finite and AE and infinite line: and this inſtance fully explains what is meant by an infinite line in this ſcience; for it means only a line longer than any determinate line which they have occaſion to take; and farther than this they give themſelves no concern. In the ninth propoſition AB and AC are infinite lines; and the point D is taken that we may have a fixt line AD and ſo in other inſtances.

I am now arrived at the concluſion of this diſſertation, in which I have endeavoured to point out ſuch circumſtances, as the thoughtleſs reader is apt to overlook. If it ſhould ſeem ſtrange to any one to

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find ſuch a cerimonious introduction to a ſimple, well defined and demonſtrative ſcience, he will be pleaſed to obſerve ſome peculiarities which diſtinguiſh this ſubject remarkably from moſt others. It is true the language is plain and ſignificant; but the truths to be communicated are conveyed in too few words to engage the attention of thoſe who have been accuſtomed to the figurative language in which moſt other ſubjects are delivered; and where the reader, inſtead of having conſequences deduced from every word, carries ſomething with him if he attend to but one word of three. And for this reaſon I have been a little diffuſe and circumlocutory; that the tranſition, from the common form of ſpeaking, to this very conciſe and accurate method of expreſſion, might be made with greater eaſe by the reader, after ſome part of the ſubject has been explained to him in his own way. But I muſt beg leave to introduce the ſtudent to Euclid himſelf, and defer all future intercourſe with him until he has reached the end of the firſt book.

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DISSERTATION III.

THE firſt book of Euclid's elements will neceſſarily ſuggeſt a great variety of reflexions to a judicious reader; he will perceive a particular method of arrangement, and a particular manner of demonſtration which makes that arrangement abſolutely neceſſary; with ſomething characteriſtic even in his way of conſtructing problems. But an indolent reader requires to be put in mind of each of theſe particulars, otherwiſe he will be diſpoſed to overlook them. Some remarks therefore upon each of theſe heads may be uſeful to put him into a proper train of thinking; and this ſhall be the ſubject of the preſent diſſertation.

CHAP. I.

In which Euclid's method of demonſtration is proved to be neceſſary contrary to the opinion of Clairaut.

"Qu'Euclide ſe donne la peine de démontrer, que deux cercles "qui ſe coupent n' ont pas le même centre, qu'un triangle ren-"fermé dans un autre, a la ſomme de ſes côtés plus petite que "celle des côtés du triangle dans lequel il eſt renfermé; on n'en "ſera pas ſurpris. Ce Géométre avoit à convaincre des Sophiſtes "obſtinés, qui ſe faiſoient gloire de ſe refuſer aux vérités les plus "évidentes: il falloit donc qu'alors la Géométrie eut, comme la "Logique, le ſecours des raiſonnemens en forme, pour fermer la "bouche à la chicane. Mais les choſes ont changé de face. Tout "raiſonnement qui tombe ſur ce que le bon ſens ſeul décide d' "avance, eſt aujourd'hui en pure perte, & n'eſt propre qu'à ob-"ſcurcir la vérité, & à dégoûter les Lecteurs."

This is a very ſingular opinion concerning the motives which led Euclid to that rigorous method of demonſtration which he has adopted: for we are here told that it was not choiſe, but the circumſtances of the times in which he lived, which brought him to write as he has done: his living among ſophiſts, drove him beyond the bounds of good ſenſe: for if he had been left to follow his own inclinations, we have here room to ſuppoſe that his demonſtrations would have appeared in a very different form, or in other words, that had he been a Frenchman and lived in the ſame polite and happy times as *Clairaut* he would have written juſt as he has done. The French would have taken his word for it, that two circles, which cut one another, could not have the ſame center; and that polite nation ſurely never would have contradicted him if he had ſaid the ſame thing of two Ellipſes: but one thing I am certain of, that, had they been acquainted with no other principles but ſuch as this author would have us appeal to, they never could have contradicted ſuch an aſſertion upon any good grounds.

Now what I mean to prove, in oppoſition to this doctrine, is, that Euclid is remarkable for adhering to the very principles which *Clairaut* thinks he himſelf has gone upon: and I ſhall now produce as many inſtances as I think the reader's patience can well endure in proof of this aſſertion, namely, that Euclid every where diſtinguiſhes himſelf by keeping clear of ſuch things, as good ſenſe would decide of itſelf before hand. Now as the method of theſe two authors is not ſimply different, but directly oppoſite the one to the other; *Clairaut* and I muſt aſſign a very different office or employment to this ſame good ſenſe, the preſuming to teach which, he ſays, anſwers no other end but to diſguſt the readers. But I am certain good ſenſe will never be either affronted or diſguſted to have any thing ſet in a better or ſtronger light than what it appeared in before; and I challange any one to produce an inſtance, where the thought is not improved and rendered more accurate by Euclid's reaſoning; and it muſt be a ſtrange kind of good ſenſe, which could reckon ſuch reaſoning, as he ſays *en pure perte*.

But as to his looking upon himſelf as bound to convince obſtinate ſophiſts, it is very obvious that he never went in the leaſt out of his way on their account; and guided himſelf by very different maxims, as will appear to every one who conſiders his manner of reaſoning, which is to the laſt degree inconſiſtent with any ſuch ſuppoſition, as that he had the leaſt intention to combat ſuch prejudices as the ſophiſts raiſe, when they have a mind to ſport with the credulity and ignorance of the multitude. It is true he furniſhes us with the means of eſcaping out of their hands, if a proper uſe be made of his principles; but it never once entered his thoughts to combat their ſilly objections, which never could be obtruded upon any one, who is not entirely ignorant of the nature of the ſubject.

However, as a full proof of what I now aſſert, I would recommend it to thoſe who may chuſe to differ from me in opinion, to read *Proclus's* commentary, who ſets himſelf in earneſt about this hopeful taſk; and there, it is granted he will find reaſonings, *qui tombe ſur ce que le bon ſens ſeul décide d'avance*; and I will allow that ſuch reaſonings are of no other uſe but to obſcure the truth, and diſguſt the readers. The diſpute therefore is brought to this ſhort iſſue; let the reader peruſe the firſt book of Euclid and then read *Proclus's* commentary, obſerving the difference between the two kinds of compoſition; and I am certain he will acquit Euclid of any deſign to convince obſtinate ſophiſts; and moreover muſt be perſuaded that he has not only omitted every thing which good ſenſe can *intuitively* determine; but that he has alſo left ſeveral things for the reader to ſupply which would not come eaſily under his own preciſe notion of demonſtration; for Horace's rule was never better nor more properly applied than by this author;

——— et quæ
Deſperat tractata niteſcere poſſe, relinquit.

Several inſtances of which, taken from his firſt book, I ſhall now proceed to enumerate, not conſidering them as overſights, but as proofs of a moſt refined and accurate judgement.

And firſt, to begin with that famous principle by which the meeting of two ſtraight lines is determined: We may readily ſuppoſe that Euclid introduced the ſeventeenth propoſition, with a view either to demonſtrate or explain the eleventh common notion, as being the converſe of it. For from this propoſition we learn that whenever the two lines meet, the angles made by the cutting line, upon that ſide, are leſs than two right angles; but we can by no means infer from this that the lines will meet whenever the angles are leſs than two right angles. But again he has demonſtrated in the twenty eighth propoſition; that when theſe angles are equal to two right angles the lines will never meet. But he cannot, from this ſay, that this is the only caſe in which the lines will never meet: only thus far he may go, that the ſuppoſition, from which he infers that they will never meet, requires abſolutely that the angles ſhould be exactly equal to two right angles; and the leaſt deviation from this ſuppoſition will render his principles of no avail; as no conſequence can follow from them; for their abſolute equality to two right angles and nothing leſs, is neceſſary to prove that the lines will never meet. Now this is ſufficient at leaſt to ground a perſuaſion, that in all other caſes they will meet; eſpecially when it is conſidered that ſuch lines are inclined towards one another, as may be ſhewn by drawing a perpendicular to one of the lines; which will prove that they are approaching towards one another upon the ſide where the angles are leſs than two right angles: and going farther from one another upon the ſide where the angles are greater than two right angles; but the opinion thus acquired is not of the nature of a demonſtration. Nor could he even ſay that a line interſecting one of two parallels, would meet the other; and indeed if this could be ſhewn upon ſcientific principles, it would be a demonſtration of the thing in queſtion: but this would be taking the thirtieth propoſition for granted, the truth of which depends upon this very principle.

We may therefore ſuppoſe that ſuch things as theſe and numberleſs others of the ſame kind muſt have occurred to a mind ſo fruitful in expedients upon ſuch occaſions, and were all rejected by him as falling infinitely ſhort of his idea of demonſtration: wherefore without perplexing his reader with impotent attempts towards a demonſtration; he judged it more proper to cut ſhort this fruitleſs ſearch by placing this principle among the common notions; and as it appears from this inveſtigation that it was not placed here without reaſon; ſo it has a right to keep its place, until it can be ſhewn that ſome abſurdity can follow from the uſe of it. And upon this point I have only to add, that it is certain from the general tenour of his demonſtrations, that if he could have exhibited it without any ſcientific defect, but in the clumſy dreſs in which ſome authors have arrayed it, he would have rejected the concluſion with diſdain, and left it where we now find it. So that it is not a demonſtration of any kind which was his object, but ſuch an one as is both elegant and forcible.

But further, I have mentioned before, what ſeems to be an exceeding good general rule of critiſciſm in this ſcience, that we ought always to examine if ſome conſequence has been drawn from every part of the ſuppoſition or conſtruction. Now there are ſeveral inſtances where Euclid has laid no ſtreſs, even, upon the moſt material part of the ſuppoſition; which ſhews how little he minded the conviction of thoſe *obſtinate ſophiſts*, for whoſe ſake according to *Clairaut* he gave his elements ſuch a formal dreſs, to the great diſguſt of every Frenchman; becauſe here would have been ſo good a pretence for ſtanding ſtill, that no arguments could have perſwaded theſe ſophiſts to proceed while ſuch blemiſhes remained. But to come to particulars.

In the twenty ſecond propoſition, it is required to make a triangle of three given ſtraight lines, but with this neceſſary limitation; that any two of them muſt be greater than the third. However no uſe is made of this limitation in the demonſtation of the propoſition; and I am perſwaded that our author omitted to prove that the two circles will cut one another, which depends upon this limitation, becauſe this proof does not admit of ſuch a natural and elegant turn, as he had determined to give to every ſtep of his demonſtrations; there being no direct principle to refer to, by which the interſection of circles can be proved; that they have ſome ſpace in common is the principle from which their interſection is to be inferred, which is indeed a common notion but not one of thoſe, which he has ſelected to draw conſequences from; concluding I ſuppoſe that all rational men, could make as ready and extenſive an uſe of ſuch notions as theſe, as he could, and therefore, that it would be unfair in him, to appropriate to himſelf either the notions or their conſequences; or according to *Clairaut*, that ſuch reaſonings would only fall upon, what good ſenſe had decided before hand, in a manner full as ſatisfactory.

Again, there is an inſtance of the ſame kind, in the twenty fourth propoſition; where the moſt material part of the ſuppoſition is never once mentioned in the demonſtration; viz. that the angle BAC is greater than the angle EDF; conſequences are tacitly drawn from it, like the taking for granted that the circles will cut one another in the twenty ſecond; for without this part of the ſuppoſition the angle EGF would not be a part of DGF, nor, if the point F was within the triangle DEG would this angle EGF be a part of the angle below the baſe of the iſoſceles triangle DFG; which is abſolutely neceſſary in order to prove that the angle EFG is greater than EGF; or, in other words, it follows from this part of the ſuppoſition, that DG always falls upon the ſame ſide of DF which it is repreſented to be upon in the figure; but it would be difficult to give this *reaſon* ſuch an elegant turn as to entitle it to a place in one of Euclid's demonſtrations.

But the ſophiſts would ſtick at much leſs matters than theſe; they would require him to prove in the firſt propoſition, that the two circles would cut one another; and that their interſection is a point: and in the ſecond that there would always be ſuch points as G and L: alſo in the fifth that AF and AE were two ſuch lines, that a part could be cut off from the one equal to the other, or that AE is greater than AF: they would likewiſe find ſeveral parts of the ſuppoſition in the ſeventh from which no conſequences are drawn, which I have taken notice of before; in the twelfth propoſition they would require him, to prove that a circle deſcribed with C for the center, and through any point on the other ſide of AB, would cut the line AB; which would give occaſion to his making uſe of the infinity of the line from which no conſequence is drawn, as the demonſtration now ſtands: They might likewiſe aſk him, upon the corollary to the fifteenth propoſition, whether when two lines meet at a point they make angles at their meeting equal to four right angles: and in the thirtieth propoſition, by what principle he makes a ſtraight line paſs through three points, that is, takes it for granted that it may paſs through three points, viz. a point in each line; for he ought to have taken two, and to have proved by the eleventh common notion that there would always be a third: Nor would it be very impertinent of them to require him to prove that AH and FG will meet in the forty fourth propoſition; and indeed in that inſtance I cannot help being of opinion that the conſtruction would have been more in Euclid's manner if he had made GH equal to BA and then joining HA had proved that HA was parallel to GB by the thirty third propoſition. In ſhort if it had been his intention to write for the conviction of the ſophiſts, his demonſtrations would have had a very different form from that in which we now find them. And it will be eaſy for any one who has read even the firſt book with attention to add a great many more to the above catalogue of omiſſions.

But I hope nobody will ſo far miſtake my meaning here as to ſuppoſe that I conſider ſuch queſtions as theſe, either as captious or improper. They are mentioned only to ſhew that Euclid had ſome plan by which he directed himſelf in his demonſtrations, very different from the ſuppoſition that it was to convince obſtinate ſophiſts. For I am ſo far from conſidering ſuch queſtions as improper; that I think they are ſuch as every one muſt aſk himſelf, and a great many to the ſame purpoſe, otherwiſe I am certain he will have a very imperfect notion of the book; and whoever is in earneſt to underſtand this ſubject ſhould take every opportunity of deviſing ſuch objections; the proper anſwers to which will be readily ſuggeſted by what Euclid has ſaid. But this ſhould be ſet about upon ſome regular plan; and I would therefore recommend it to the ſtudent, after he has finiſhed the firſt book to give it a ſecond reading, examining every ſuppoſition and conſtruction, and the conſequences drawn from them; and their full importance in the propoſition; which is not to be determined from the number of words in which they are expreſſed, but by the dependence which the concluſion has upon them: for I have an inſtance juſt now in my view, in which I may ſafely ſay that the firſt reading hardly ever diſcovers the full importance of the ſuppoſition: what I mean is the ſuppoſition in the forty ſeventh propoſition; from which the conſequence that follows is; CA and AG as alſo BA and AH make but one ſtraight line. Now whoever has read this propoſition without perceiving that the concluſion depends entirely upon this, cannot be ſaid to have underſtood the demonſtration; and that this is true appears; becauſe the ſquare BG is only double of the triangle BFC becauſe GAC is a ſtraight line. The reader is alſo carefully to obſerve whether the concluſions are general or particular; becauſe it may ſo happen that a particular poſition of lines or points will lead to a concluſion, which will by no means follow if theſe are changed; and often if it do hold good it may require ſome new ſteps to come at the concluſion. But it ſeems neceſſary to illuſtrate this by particular examples.

In the eleventh propoſition of this firſt book, it is required to draw a perpendicular to a line, from a point given in it. This may be divided into two caſes; for the point may be at ſome diſtance from the extremity, or it may be the extremity of the line. Euclid has conſidered the firſt caſe only: and there is but one falſe conſequence which could be drawn, by taking the other for granted; and what is very remarkable *Simſon* in his edition has hit upon this very conſequence, in attempting to prove that two ſtraight lines cannot have a common ſegment; becauſe it muſt be taken for granted that two ſtraight lines cannot have a common ſegment before a perpendicular can be drawn from the extremity of the line; for as the line muſt be produced, without this limitation, it may take two directions.

Again in the twenty fourth propoſition, the point F may have three different poſitions; it may be without the triangle DEG or in the line EG or laſtly within the triangle: For a particular inſtance the ſtudent may make ſuch a triangle as ABC in the figure to this propoſition; having AB longer than AC; and after the triangle DEG is made; with the center D and diſtance DG deſcribe a circle; which will cut EG; now any ſtraight line drawn to the circumference of this circle, from the point D, within the angle EDG, may repreſent the line DF; from which the three different poſitions will be obvious. When the point F falls within the triangle, the concluſion may be inferred from the twenty firſt propoſition; but it would be more uniform to produce DF and DG; and to reaſon upon the angles below the baſe, in the ſame manner as Euclid has done upon thoſe that are above it. It is true *Simſon* reduces this to one caſe, but I think not in the manner of Euclid;

for there is no circumſtance given by which we can determine which ſide is the longeſt; there ought therefore to be added to the preſent ſuppoſition, *AC being longer than AB*; but the determination of this point I ſhall leave to the curious. But farther I would even recommend the ſame kind of examination, where there was no probability of finding any variation; as for inſtance in the ſixteenth propoſition, the concluſion depends upon FCD's being an angle; for if that could ever ceaſe to be an angle, the outward angle might be equal to the inward oppoſite one; but that is impoſſible becauſe then the point F would be in BC produced; and conſequently the point E upon C that is the middle of a line at its extremity, which is abſurd.

Having been thus particular in ſhewing what is not Euclid's method of demonſtrating, I ſhall conclude this chapter with a few inſtances tending more directly to explain what it is.

And his great peculiarity ſeems to be a determined reſolution always to refer directly to ſome principle, and never truſt to a vague conception; or more properly never to make uſe of a vague expreſſion: The thirteenth propoſition is a remarkable inſtance to this purpoſe; That the angles are equal to two right angles ſtrikes the ſenſes immediately and produces a conviction which has ſomething fluctuating in the nature of it; ariſing from obſerving that the angle ABD is above a right angle, by juſt as much as ABC is leſs than one; now this is not a principle ſufficiently diſtinct to reaſon from in a demonſtrative ſcience; and Euclid has ſhewn great art in the demonſtration of this very propoſition; which he has reduced to the common notion, *magnitudes which are equal to the ſame are equal to one another*; and how accurately he keeps up, through the whole demonſtration, to the notion of the angles being magnitudes, cannot fail to engage the attention of a judicious reader. The twentieth propoſition furniſhes an inſtance to the ſame purpoſe: every one believes that two ſides of a triangle are greater than the third; and he may take up this opinion from a conſideration of many different circumſtances; he may conſider that a ſtraight line muſt ſurely be the ſhorteſt way between two points: or he may truſt to the judgement of the aſs of the *Epicureans*; when a bundle of hay is placed at one of the angles; but ſtill he will find ſomething unſettled in his conviction as reſting upon no determinate principles; and Euclid never leaves any thing which he takes in hand to demonſtrate, in this unſettled ſtate: he makes this property follow, not from any unſtable principles or random conceptions, but from the very nature of a triangle; and a thorough examination, of the whole appartus neceſſary for the demonſtration of this principle, will give the intelligent reader a wonderful inſight into Euclid's method of demonſtration.

It ſeems almoſt a deſperate undertaking, to endeavour at reviving a taſte for accurate demonſtration; becauſe the common vague conceptions of magnitude, proped by a ruler and compaſſes, when they begin to totter, are judged capable of performing every thing that can be expected from this ſcience. But if a reformation is at all practicable, it ſeems moſt likely to be brought about, by turning the attention of the world to Euclid's method of demonſtration; and I ground my hopes of ſucceſs upon two circumſtances; namely that the modern writers of elements of Geometry neither underſtand the nature of their own demonſtrations, nor the force of thoſe of Euclid. Becauſe if they underſtood the force of Euclid's they would be aſhamed to dignify their own with ſuch a name; and if they underſtood the nature of their own, they muſt perceive that a great deal of expence both of time and thought might be ſaved by delivering their propoſitions as facts, which had been demonſtrated; unleſs indeed we ſuppoſe that what they call demonſtrations are added merely in compliance to vulgar prejudice. Now could a judicious reader, no ways intereſted in the iſſue of the debate, be once brought to weigh the merits of ſuch demonſtrations; he muſt either decide in favour of ſuch as Euclid's, or declare that all demonſtration was now become unneceſſary; and that therefore the farce of it ſhould be laid aſide. But upon ſuch an event, ſuch is the charms of truth, I would naturally expect a conſiderable number to leave their new maſters, as ſoon as they had relinquiſhed their pretenſions to demonſtration: and it is upon ſuch hopes only that I proceed to give ſome other inſtances as characteriſtical of Euclid's method of demonſtration.

Euclid's demonſtration of the ſeventh propoſition, may be compared with the one given in *Deſchales's* edition of Euclid; which was no doubt intended as an improvement in ſome reſpect or other. And this ſeems to me the more extraordinary becauſe the domonſtration of the ſeventh is remarkably diſtinct and pertinent; leading to the concluſion ſo directly that the full force of the demonſtration muſt be perceived by the mind: and I am perſuaded that our author would have given up all thoughts of writing his elements, if he had found himſelf obliged to reſt ſuch a material part of his ſcience upon ſo unſtable a foundation, as the deſcription of two circles, with an indeterminate radius and even before it could be known in how many points their circumferences would cut each other; becauſe every point of interſection, that did not lie on the other ſide of AB would contradict the propoſition; and prove that two triangles might be placed according to the ſuppoſition, until theſe points were reduced to one, which is not ſo eaſy to do as a ſuperficial reader may imagine. But there is nothing which I have obſerved Euclid more cautiouſly to ſhun than the deſcribing a circle with an indeterminate radius; or more properly, than the fixing a line, otherwiſe undetermined, for the purpoſe of deſcribing a circle. It is true he makes uſe of conſequences which muſt be derived from circles deſcribed in this manner; but this is ſuppoſed to be tranſacted in ſuch a manner as to bring no diſgrace upon the ſcience. In the ſixteenth propoſition AC the ſide of any triangle is required to be cut in halves, which cannot be done until the line is fixed; but then I immediately perceive that, notwithſtanding this, as no conſequence is deduced from AC's having any particular length, the demonſtration is nevertheleſs general; ſo that I can finiſh my conſtruction and have AC as undetermined as I found it; ſo that this tranſaction is ſuppoſed to be carried on in private by the reader himſelf; and which he acquieſces in, as bringing no reflexion upon the ſcience.

There is another thing which I ſhall juſt obſerve before finiſhing this chapter: Euclid's demonſtrations are often more general than they ſeem to be; for whenever, it would be only a repetition of the ſame ſteps, he always omits them, whether in a conſtruction or demonſtration. The forty fifth propoſition furniſhes a remarkable inſtance of this; he ſhews how to turn a fourſided figure into a parallelogram &c. and as no new circumſtance would occur, whatever be the number of ſides of the figure; he concludes that it is general, without adding a word more; which has miſled ſome ignorant conceited people to think that they improve upon Euclid by ſhewing a ſimpler way of turning a four-ſided figure into a parallelogram; never conſidering that Euclid has turned any rectilineal figure into a parallelogram, whatever be the number of its ſides; and whatever be their poſition to one another. Now I hope it is evident that Euclid had ſome other plan in his demonſtrations than to convince the obſtinate ſophiſts; and what that plan is I ſhall explain in the next chapter.

CHAP. II.

Of the arrangement of Euclid's propoſitions.

THOSE who know nothing of geometry tell us the ſcience ſhould be delivered according to the following arrangement. We ſhould firſt begin with *a point*; and after having laid before the reader all its various properties, ſuch as poſition, want of dimenſions and the like, we are next to proceed to the ſtraight line; and after delivering every thing that can be ſaid of a ſingle ſtraight line, we may then advance a ſtep farther and take two of them under our conſideration; firſt enquiring into the properties of ſuch as are parallel; and this ſubject being exhauſted we may next examine two ſtraight lines that meet ſo as to form an angle; which will furniſh us with a fruitful ſource of ſpeculation; becauſe this will now be the proper place to treat of angles; to ſhew how they may be proved to be equal, and into how many kinds they ought to be divided, not forgetting to prove that a right angle conſiſts of ninety degrees. Now ſuppoſing the properties of ſtraight lines exhauſted, the natural tranſition is to ſuch as are crooked, arranging them into different orders, beginning with the circle; but take care not to conſider it as a figure; for that would deſtroy all order; for it is the circumference only that belongs to our preſent ſubject: we may next proceed to the conic ſections, and ſo on to other lines.

The buſineſs of lines being finiſhed, there would be abundant matter for diſpute, when we came to treat of figures, whether we ſhould begin with the triangle or the circle; with the triangle as being the ſimpleſt rectilineal figure; or with the circle as being bounded by one ſingle line; the circle might probably have the preference upon a full examination as being a *perfect figure*, and for other reaſons which it would be too tedious to inſiſt upon at preſent. But ſuppoſing this difficulty got over; the triangle would certainly claim the next place; and of all the different kinds of triangles, the equilateral would put in for the firſt; the iſoſceles would deſerve to be conſidered next; from the iſoſceles the tranſition muſt be to the ſcalene; taking care all the time, in delivering their properties to ſay nothing of their angles, for that would be to confound the nature of things; becauſe they are afterwards to be divided according to their angles, right, acute and obtuſe; and this new diviſion would furniſh the proper place for that part of the ſubject. From triangles the moſt natural progreſs would be to the ſquare, and then to figures leſs regular. And this is the true philoſophical arrangement.

Such are the engines which *Folly* has been erecting for battering this noble work of our author; ſeconded with other auxiliary ſchemes, the force of which ſeems leveled againſt it, though more indirectly. But what holds the greateſt vogue at preſent, as falling in with our indolent diſpoſition, and bids fair for driving him to the laſt extremity, is the propoſal of conſidering no more of the ſubject than what is abſolutely neceſſary; with the inviting title; *as applied to ſuch and ſuch uſeful purpoſes of life*. Elements of geometry carefully weeded of every propoſition tending to demonſtrate another; all lying ſo handy, that you may pick and chuſe without ceremony. *This is uſeful in fortification*: you cannot play at billiards without this. You only look through *a teleſcope* like a *Hottentot* until this propoſition is read; with many ſuch powerful ſtrokes of Rhetoric to the ſame purpoſe. And upon ſuch terms, and with ſuch inducements who would not be *a mathematician*? Who would go to work with all that *apparatus* which I have deſcribed as neceſſary for underſtanding Euclid; when he has only to take a pleaſing walk with *Clairaut* upon the flowery banks of ſome delightful river, and there ſee with his own eyes, that he muſt learn to draw a perpendicular, before he can tell how broad it is: ſuch is the ingenuity of theſe Frenchmen,

> Ut puerorum ætas improvida ludificetur,
> Labrorum tenus.

I had formerly the ambition to join myſelf to this fraternity, and taking a walk in the *french ſtile* along the banks of a river, propoſed to teach the reader, how to meaſure its breadth by his hat; but I ſoon involved him ſo much in the properties of triangles, that I diſpaired of making my point good, in the true elegant and undemonſtrative manner; and ſo was obliged to get into the old beaten road as faſt as poſſible.

But there is another rank of geometricians, between thoſe, I have been ſpeaking of, and Euclid: whom for diſtinction ſake I ſhall call the *omnipotent*. Theſe ſetting out with the ſuppoſition, that they can do every thing, undertake to demonſtrate every thing. An *omnipotent* geometrician makes nothing of demonſtrating the fifth propoſition, for it is only ſuppoſing that he has cut the angle at the vertex in halves and the thing is done; never conſidering that the doing of this, requires it to be firſt proved that the angles at the baſe of an iſoſceles triangle are equal to one another. The only inconvenience of this method is, that ſhould the author's omnipotency be once called in queſtion; his whole fabric muſt be leveled with the ground.

But to return to *Clairaut*; I would have nobody imagine that I entertain a mean opinion of his abilities, from any thing that has been ſaid: we only happen to differ as to the proper manner of communicating mathematical knowledge; which I contend ſhould be either as facts or as demonſtrations; and that there is no middle way; being perſuaded that the mixing a kind of theory with the practice has been of bad conſequence to both; as this has been the occaſion of putting very improper perſons into employments, by which many fine opportunities of making improvements have been loſt to the world.

But farther *Clairaut's* method is even unnatural; for the ſcience never could have been invented according to his plan; becauſe it impoſſible that human reaſon, unleſs unnaturally confined, would be kept floating upon the ſurface of things: it is of the nature of man, when left to himſelf, to ſeek a ſolid foundation. His book, it is true, will make a ſuperficial reaſoner; but mankind left to themſelves, if they did ſo much, would neceſſarily do a great deal more.

And I now proceed to give an account of what ſeems to me a much more natural courſe for them to have taken in proſecuting their diſcoveries in this ſcience; which will at the ſame time explain the nature of Euclid's arrangement, and obviate every objection which has been made to it: but this ſhall be the ſubject of the next chapter.

CHAP. III.

The ſame ſubject continued.

It will be found to be a miſtake to ſuppoſe that we naturally begin to examine the ſimpleſt things firſt; for things are neither ſaid nor done in the ſimpleſt manner by the firſt adventurers. Our attention is more like to be drawn to a ſubject by ſome particular circumſtances, which made the conſideration of it neceſſary, than by any determination of the mind to make it a matter of contemplation; and by coming upon us thus, as it were unawares, our underſtanding is overpowered, when we undertake the examination of a thing encumbered with ſo many additional parts, which have no connexion with the ſubject in queſtion; and ſuch as it will require leiſure, ſkill and experience to ſeparate from the others.

We are not therefore to imagine that the examination, either of *a line* or *a point* would be the firſt attempts for ſetling mens notions with regard to magnitude. It is more reaſonable to ſuppoſe that the firſt kind of magnitude which would fall under the conſideration of men, circumſtanced as we are in ſociety, would be rectilineal figures, and perhaps circles or even ſolids of ſome determinate ſhapes: not only becauſe nature produces ſuch a variety of them ready made to our hands, thus, as it were, inviting us to make them the ſubject of our mediation, but ſo ſoon as property in land became eſtabliſhed, a man had only to diſcover that a greater quantity of it would yield a greater increaſe, to engage him to pay attention to this quantity; either that he might be the better able to impoſe upon his neighbour, or to guard himſelf from the impoſitions of others.

And although they might at firſt imagine that the bounding lines would be ſufficient for aſcertaining their property; yet they would come at laſt by reaſoning and blundering, to perceive that the ſhape is neceſſary to be conſidered, as well as the bounding lines. This might lead them to make pictures or reſemblances of their fields, by way of aſcertaining their property, until by freſh experiments and blunders, it may be ſuppoſed they could perceive, that theſe repreſentations, which I imagine to be made by the eye, without the aſſiſtance of other principles or inſtruments than ſuch as occaſionally offered themſelves, were alſo inaccurate. Or even if this could have determined one's property, when joined to other circumſtances upon the ſpot; yet ſtill a very material defect would be obvious; as there would nevertheleſs be wanting, any thing like an accurate way of comparing different figures with one another, eſpecially when their ſides are numerous and their ſhapes unlike. And it is probable that they would at firſt drop all thought of being able to compare ſuch with any degree of accuracy, but giving them up as unmanageable, would bend their whole attention to the ſimpleſt kind, very probably to the triangle; until the great ſtep was taken, which was to lay open the whole myſtery of this ſcience, by the man who firſt diſcovered that all rectilineal figures may be divided into triangles; and this ſtep once taken, they would then begin to tread on ſcientific ground. They would be made ſenſible of the uſe of the triangle, and exhauſt all their thoughts and ingenuity to diſcover its properties.

But it ſhould be obſerved, becauſe I believe it to be the great ſpur to improvement, that there would probably appear great inequality in the accuracy of the different concluſions which their ingenuity would lead them to, when their curioſity was once awakened by ſo fair a proſpect of being able to maſter their ſubject: ſome amounting to abſolute demonſtration, while a great many things perhaps fell ſhort of ſuch an evidence as was neceſſary to perſuade them that the concluſions were true; and ſuch diſcordant parts would neceſſarily awaken their induſtry to give the whole an uniform appearance. It would be ſeen that if ſuch a thing were proved ſuch another would follow; and they would collect as many of theſe conſequences as they could relating to triangles; what would prove them to be equal or unequal; until they came to perceive the remarkable utility of this wonderful figure, and its aptitude to explain and connect the ſimpleſt and moſt intricate parts of the ſcience.

And this ſeems to me to be the moſt natural account of the original and improvement of geometry; becauſe I am perſuaded that if any thing, *even* uniformly inaccurate, had been produced by themſelves; or if the regular mechanical rules ſufficient for the ordinary purpoſes of life had been communicated to them by ſome ſuperiour Being; mankind would have never thought of any farther improvement; and the human race would have drudged on in the ſame beaten path void of curioſity and fully ſatisfied with their preſent acquiſitions. But this we find to be a fact by experience; for thoſe who are inſtructed only in the practical parts of this ſcience, proceed with as little hopes or deſire of improvement, as the very inſtruments with which they work.

But to return to the ſubject; let us ſuppoſe at firſt that they had it only in view to give irregular figures a more regular ſhape; or in other words, that they wanted to perform the problem, which Euclid has given in the forty fifth propoſition of his firſt book; namely, to turn any rectilineal figure into a parallelogram having a given angle: it will be found that almoſt every propoſition which goes before, is neceſſary towards a proper ſolution of this problem, and if the ſtudent would be at the pains to examine how far this is true in fact, it will aſſiſt him much in comprehending Euclid's arrangement; eſpecially if he attend particularly to the uſe which is made of the triangle: and with a few remarks upon this I ſhall conclude the chapter.

And firſt it is to be obſerved that ſtraight lines and angles are too ſimple of themſelves to bear a ſtrict examination, with any hopes of diſcovering their properties; and if we want to prove them to be either equal or unequal; the lines muſt be ſides of a triangle and the angles are angles of ſome triangle; the fourth, fifth, ſixth, ſeventh and almoſt every propoſition in the book furniſhes us with inſtances to this purpoſe; and it is owing to this ſure guide deſerting us that, we are brought into ſuch difficulties by the eleventh common notion. It is true that by the aſſiſtance of a perpendicular, we ſeem to make a few ſteps in comparing angles which do not belong to any triangle, as in the thirteenth, fourteenth and fifteenth propoſitions, but as theſe depend upon the perpendicular, they are virtually ſupported by the triangle: and the ſame may be ſaid where parallel lines are concerned. That the lines from the center to the circumference of a circle are equal, is the only principle by which lines can be proved to be equal independent of the triangle. Secondly we may obſerve how the tranſition is made from triangles to fourſided figures; and the triangular ſpaces compared with parallelograms, by which a foundation is laid for turning all rectilineal figures, however irregular into parallelograms. And laſtly when the uſefulneſs of the triangle is conſidered, the reader will not be ſurprized to find ſo much pains taken, to ſettle every circumſtance which can prove them to be equal, and in what caſes ſome of their parts may be equal and the others not.

CHAP. IV.

Containing ſome remarks on the conſtructions in the firſt book.

IT has been remarked already that our author diſcovers great caution in the uſe of the poſtulates, by keeping all mechanical inſtruments as much as poſſible concealed from public view: and indeed he ſeems almoſt aſhamed of them, for which reaſon he never conſtructs his problems upon a ſuppoſition that you make uſe of any particular inſtruments; and this is neceſſary to be obſerved before we can form a proper judgement of the ſimplicity of his conſtructions. The common modern way of judging upon this point, proceeds upon a very mechanical principle indeed; for the ſimplicity of a conſtruction is to be determined by the number of times which you have occaſion to open your compaſſes: but as Euclid

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Euclid knows nothing of a pair of compaſſes, it can never be ſuppoſed that this is the teſt to which he appeals; but to the nature and number of ideas ſuggeſted to the mind by the different ſteps of the conſtructions. He does not care how often you are obliged to open your compaſſes, provided the perceptions, conveyed to the mind by his conſtructions, be the moſt ſimple and general which the circumſtances of the caſe will admit of. Some particular inſtances will explain my meaning.

In the twenty third propoſition we are required to make one angle equal to another; and you ſave yourſelf the trouble of one opening of your compaſſes by making DEF an iſoſceles triangle: but Euclid requires you only to join EF; which certainly is a ſimpler conception, than when you are moreover required to take care that your other two ſides be equal; and has alſo this great ſcientific advantage, that it is more general, by ſhewing that any triangle will anſwer the purpoſe; when your mechanical conſtruction goes upon the ſuppoſition that a particular kind of triangle is neceſſary. Thus it often happens that our compaſſes and underſtandings are at variance; or rather our hand and head incline different ways; what is moſt intelligible to the head, the hand finds the greateſt difficulty in performing.

In the ſame manner, in the ninth propoſition, three circles muſt be deſcribed according to Euclid's plan, whereas mechanically, which ſuppoſes that we can deſcribe an iſoſceles triangle upon a given ſtraight line, one opening of the compaſſes will anſwer the purpoſe; and there is a like difference in the ſcientifical and mechanical conſtructions of the tenth, eleventh and twelfth propoſitions.

The drawing of parallel lines Euclid uniformly refers to one principle; and this problem is performed by making the alternate angles equal; and this the nature of the ſcience requires that he ſhould do: however I have no objection to the uſe of any inſtrument which may anſwer the purpoſe, provided the ſtudent know what he is doing, and do not work at random without any ſettled principles.

Parallel rulers are of all contrivances, the moſt inconvenient for this purpoſe of drawing parallel lines; but yet it might be of uſe for the ſtudent to conſider upon what principles we pretend to uſe them: and he will find that they are conſtructed upon geometrical principles, though not by any propoſition in Euclid; for their parallelism depends upon the converſe of the thirty fourth propoſition, namely, that if the oppoſite ſides of a quadrilateral figure be equal, they will be parallel; the demonſtration of which will be obvious to the attentive reader; and he might at the ſame time conſider the converſe of the other part of the ſame propoſition, viz. that if the oppoſite angles of a quadrilateral figure be equal, the figure is a parallelogram; this follows directly from this conſideration; the oppoſite angles being equal to one another, and the four angles equal to four right angles, any two of them taken in order round the figure will be the half of the four right angles; that is equal to two right angles, and therefore the lines are parallel.

But the moſt convenient inſtrument for drawing parallel lines, as alſo a perpendicular, is a triangular piece of wood in the ſhape of a right angled triangle: for, by making this ſlide along the edge of a ruler, parallel lines may be drawn, upon the principle that the outward angle, is equal to the inward &c. by the twenty eight propoſition. And if you have two parallel lines given you, a parallelogram may be formed by the thirty third propoſition, by taking two equal parts of them, and joining their extremities which are towards the ſame parts; in ſhort I would recommend every kind of conſtruction which is derived from ſcientific principles, provided the reader give himſelf the trouble to find out the principles.

It ſometimes happens that a direct conſtruction will not anſwer the purpoſe; of which the amendment propoſed by me in the conſtruction of the forty fourth propoſition is an inſtance; and if any one try to demonſtrate the forty ſeventh propoſition, by deſcribing the ſquare upon the ſame ſide of BC upon which the triangle ABC ſtands; he will meet with another inſtance of the ſame kind; for if he deſcribe this ſquare, he will find it no eaſy matter to prove that its ſides are terminated in FG and KH or in theſe lines produced, which is abſolutely neceſſary to make good the demonſtration: but if he draw perpendiculars from the points B and C terminated in theſe lines and joining their terminations, it is eaſy to prove the figure to be a ſquare; ſuch indirect conſtructions ſhould be particularly noted by the learner wherever they occur. In ſhort to conclude what I have to ſay about conſtructions, I would have the ſtudent exerciſe himſelf, in them upon every occaſion, or even to ſeek occaſions for doing it, only with the caution to beware of contracting ſuch prejudices as might corrupt his ſcientific principles, which can never be the caſe, if he know what he is about, and conſider the nature and extent of his inſtruments.

CHAP. V.

Containing ſuch remarks as may enable the reader to make a proper eſtimate of his progreſs in geometry when he has made himſelf maſter of this firſt book.

BEFORE he proceeds to the ſecond book it will be proper for the reader to take a review of this firſt, not as a learner, but in order to make a proper eſtimate of his geometrical acquiſitions; and he will not be much out in his reckoning if he judge of the whole by what he knows of the triangle. I have mentioned already that this figure is our great inſtrument of diſcovery, being equally neceſſary for determining the ſimpleſt properties of ſtraight lines, and angles, as thoſe of the moſt irregular figures.

The firſt general property which we learn of this figure is that all triangles are equal, which have two ſides equal to two ſides each to each, and the angles contained by the two ſides equal; and that this equality does not ſimply hold with reſpect to the ſpaces, but is true of every ſingle part of the triangle; their baſes are equal, the two remaining angles in each are equal, viz. thoſe under which the equal ſides are extended, each to each: or in other words that the two ſides, and the angle between them, fixes every part of the triangle without a poſſibility of change. The fifth and ſixth propoſitions contain more partial properties of the triangle; by proving that the equality of the two ſides will always inſure the equality of the two angles at the baſe: and converſly that the equality of the two angles at the baſe fixes the equality of the ſides. Thus we know what will prove the triangle to be iſoſceles; and we know the moſt remarkable conſequences which follow from its being ſuch a triangle.

In the eighth propoſition it is demonſtrated that all triangles which have their three ſides equal, each to each, are equal in every reſpect: or in other words, that the three ſides fixes the triangle, ſo that neither its ſpace or angles can poſſibly vary ſo long as the ſides continue the ſame: and in the twenty ſixth propoſition we have two other teſts to the ſame purpoſe: for if two angles and the ſide between them be given or fixt we cannot change any part of the triangle, neither the other two ſides nor the other angle; ſo that all triangles which have theſe equal muſt be equal in every reſpect: or ſo long as two angles of a triangle and a ſide extended under one of them are fixt the whole triangle is fixt; therefore all triangles which agree in theſe particulars muſt be the ſame or equal in every reſpect.

Theſe four teſts which fix the triangles abſolutely ſhould be carefully examined not only ſeparately, but compared together before the reader can make the proper uſe of them.

It is not only neceſſary to obſerve in what circumſtances the triangles may be demonſtrated to be equal; but it is of no leſs importance to ſettle in what caſes, we can ſay, that the whole triangles or certain parts of them muſt be unequal. And firſt it appears, from the ſeventh propoſition, that if the two triangles be upon the ſame baſe and upon the ſame ſide of it; their ſides which are terminated in the ſame point cannot be equal. Alſo it follows from the twenty firſt propoſition, that if the triangles be upon the ſame baſe, and the vertex of the one triangle within the other; their ſides cannot be equal, either each to each or taken both together; but that the ſides of the included triangle will be leſs than the two ſides of the other.

Such concluſions as theſe indeed, we are both very dexterous and confident in drawing at random; and the following would appear ſtrickingly concluſive; becauſe it it proved in the fourth propoſition, that, when the two ſides are equal, each to each, and alſo the angle contained by the two ſides, therefore the baſe is equal to the baſe; therefore undoubtedly we may infer from this, that if the included angles are not equal, neither will the baſes be equal, but the one muſt be longer than the other; and certainly the larger angle ought, by the rule of right, to have the longer baſe extended under it. But ſoftly, good ſir, this is not geometrical reaſoning: for although the equality and coincidence of the baſes is demonſtrated in this propoſition when you keep up to the different parts of the ſuppoſition; yet it is not ſhewn that this is the only caſe in which that equality can happen: and indeed the changing a ſingle circumſtance will not prevent their equality for you may ſuppoſe AB to be unequal to DE and nevertheleſs the baſes may be equal. So that however ready a ſuperficial reader may be to adopt ſuch concluſions, yet the geometrician will find that a great many ſteps are to be taken before he can demonſtrate this to be true. Which is indeed done in the twenty fourth propoſition, but after many neceſſary properties have been previouſly delivered. If the two ſides are equal but the baſes unequal it appears from the twenty fifth propoſition, that the angle oppoſite to the greater baſe is the greater.

But farther, with reſpect to the angle: I have mentioned already, that we cannot compare angles unleſs we conſider them as belonging to triangles; the ſixteenth propoſition is made uſe of to prove that one angle is greater than another: which is a very fundamental propoſition and ought to be carefully examined: the ſimpleſt view which it is poſſible to take of it ſeems to me to be this; to conſider the propoſition as expreſt thus; the angle ACD is greater than CAB and alſo the angle BCG is greater than ABC; which when properly conſidered will appear to be the ſame thing: and then, when we recollect that the angle ACD is equal to BCG, it may certainly be ſaid that ACD is greater than either CAB or ABC: but this is a propoſition which ought to be viewed in every light. By this propoſition we likewiſe prove that lines in the ſituation, which the ſuppoſition requires in the twenty ſeventh can neve rform a triangle, and thus leads us to parallelograms; from the conſideration of which we have a new and wonderful property of triangular ſpaces, namely, that if the baſe of the triangle continue the ſame or equal, you may vary the ſides until they become a thouſand times longer than in their firſt ſituation, and ſtill the triangular ſpace will be the ſame. And this is alſo the principle by which we can change ſpaces from one ſhape to another as will be obvious to every one who underſtands the forty fifth propoſition.

I ſhall conclude this diſſertation, with a general admonition to the reader, to obſerve particularly the uſe of the ſuppoſition in every propoſition; becauſe I know nothing that is ſo eaſily neglected: the conſtruſtion and even the references to other propoſitions may make a forcible impreſſion upon the mind; but the ſuppoſition has hardly any thing which can awaken our attention to it, except a want of evidence in the concluſion, and there are thoſe who have a wonderful inclination to lay aſide their ſcruples at that part of a demonſtration.

DISSERTATION IV.

WHOEVER has read the firſt book with care, will perceive the particular uſe of the triangle for diſcovering and connecting the different properties of magnitudes; and muſt at the ſame time be ſenſible of the uncommon genius of the author, who could apply this figure in ſuch a manner as to make it diſcover its own properties, as well as thoſe of other magnitudes. In his ſecond book he preſents us with a new inſtrument, the uſe of which is no leſs extenſive than that of the triangle; and it is the purpoſe of this diſſertation to draw the attention of the ſuperficial reader to the properties and moſt obvious uſes of the rectangle.

CHAP. I.

Of parallelograms.

TO underſtand this ſecond book properly, it will be neceſſary to conſider the thirty fifth propoſition of the firſt book, in more points of view than are immediately preſented to us by the propoſition itſelf. It is there demonſtrated that when the parallelograms are upon the ſame baſe, and between the ſame parallels, the parallelogram ſpaces are always equal: now this reduces an infinite variety of parallelograms to one ſingle parallelogram as far as the ſpace is concerned; and if a choiſe of a parallelogram is to be made, which is to repreſent all the others, the rectangle, or right-angled parallelogram ought to have the preference; becauſe its bounding lines are the leaſt, and its angle fixt or determinate; that its ſides are the leaſt will be obvious, by comparing a ſide of the rectangle with the ſide of any other of this infinite variety of parallelograms juſt mentioned; as for inſtance AB with BE, the one of which ſubtends an acute angle and the other a right angle in the ſame triangle.

But this rectangle itſelf, is ſufficiently or rather abſolutely determined by any two of the lines about any one of its angles: and thus two ſtraight lines, will fix an infinite variety of parallelogram ſpaces.

Now this is the firſt definition of the ſecond book, which is thus expreſſed; Every right angled parallelogram, is ſaid to be contained by any two of the ſtraight lines containing the right angle. The ſpace indeed is bounded by four ſtraight lines; but not by four different ſtraight lines, for the oppoſite ſides are the ſame, or equal. Here the reader is carefully to obſerve that there is no difference made here between *equality* and *identity*; for whatever lines are equal, in the preſent caſe they are always conſidered as the ſame; that is if you make a right angled parallelogram, any two ſtraight lines whatever, that are equal to the two lines about one of the angles; are ſaid to contain the ſpace, as well as the particular lines which make two ſides of the very figure.

This circumſtance perplexes beginners exceedingly; and ought to be made the ſubject of frequent meditation before one attempts to read a ſingle propoſition in this book. And here it would be proper to begin with conſidering the nature of a parallelogram, ſtill more particularly: the reader has ſeen what will fix the parallelogram ſpace: he is next to conſider what will fix the whole parallelogram, ſides and angles; and he will firſt obſerve that the two ſides about any of the angles; will determine all the ſides; becauſe the other two ſides are equal to theſe two; next let him try to find out what will determine the angles. Now any angle of a parallelogram being given, it is impoſſible to change any of the others; but this the reader will do well to demonſtrate; which he may do as follows; any two angles of a parallelogram which follow each other in order are equal to two right angles; one of theſe being fixed ſurely the other is fixed alſo; and the oppoſite angles of parallelograms are equal; therefore it is obvious that one angle of the parallelogram fixes all the others. The reader ſhould demonſtrate this formally and then he will ſee that the parallelogram is entirely fixed when an angle, and the two ſides about that angle are given. But as I know from experience that the ſtudent will not perceive this immediately; let him take only the two ſides, and the angle between them; and with theſe compleat a variety of parallelograms; and he will always find them the ſame. Not that this conſtruction is neceſſary towards a right underſtanding of the concluſion; but I know that it is neceſſary for fixing the reader's attention; and without attention it is impoſſible to comprehend any concluſion. Now this point being once ſettled, it is very clear, that if the parallelogram be right angled, the two ſides about one of the angles, or even any lines, equal to theſe, will fix the parallelegram; from which the full import of this definition may be underſtood.

CHAP. II.

Containing ſome remarks on the principles made uſe of in demonſtrating the firſt eight propoſitions.

THE ſimplicity of the demonſtrations made uſe of in the firſt eight propoſitions, will be readily acknowledged by every one who underſtands them: and yet it often happens that the ſtudent is at a loſs to comprehend their meaning, eſpecially upon the firſt reading; which is owing to two circumſtances; firſt becauſe he does not underſtand the firſt definition; and ſecondly becauſe we have no common notions of the properties of figures contained in this book; if ſome property of a triangle ſhould be accidentally mentioned to a perſon ignorant of the principles of this ſcience, he would nevertheleſs form ſome opinion concerning its truth or falſhood from ſome common notions of his own; but read the enunciation of the third propoſition in this book to him; and he will have no more opinion of its truth or falſhood than if it were delivered to him in an unknown tongue. And ſuppoſing theſe two difficulties to be got over; that is ſuppoſing the firſt definition to be well underſtood, and the meaning of the propoſition become familiar, it is impoſſible for any evidence to preſent itſelf to the mind in a ſimpler form, than that which theſe demonſtrations offer; as the progreſs of it in general is thus; you deſcribe a certain rectangle or ſquare, which is divided into right angled parallelograms or ſquares by lines drawn parallel to the ſides; and the whole myſtery conſiſts in ſettling what theſe ſpaces are according to the firſt definition. For inſtance, in the firſt propoſition, as the whole is equal to its parts, the rectangle, or right angled parallelogram, BH is equal to the rectangles BK, DL, EH taken all together; ſo that we have only to ſettle what the rectangles BH, BK, DL, EH are according to the firſt definition, and the demonſtration is compleat. In the ſame manner in the ſecond propoſition (which by the bye is only a particular caſe of the firſt,) the ſquare ABED is made up of the two rectangles AF and CE; ſo that we have only to determine what theſe are according to the firſt definition, and then the thing propoſed is demonſtrated. Again in the third propoſition, the rectangle AE is made up of the rectangle AD and the ſquare CE; ſo that we have only to determine what theſe are according to the ſame firſt definition, and the propoſition is demonſtrated and ſo in other inſtances.

The firſt eight propoſitions ſeem to a ſuperficial reaſoner to reſt in ſome meaſure upon a kind of intuitive evidence; becauſe the form of demonſtration differs remarkably from that made uſe of in the remaining propoſitions of this book; in which the concluſions are inferred from the forty ſeventh propoſition of the firſt book without exhibiting the ſquares. But Euclid never draws any conſequence from what we are ſuppoſed to ſee, though we ourſelves may: for his demonſtrations are equally concluſive whether the figures upon which we reaſon be the objects of our ſenſes or not. The different ſpaces exhibited in theſe figures ariſe from the conſtruction, and have their exiſtence and properties from that, whether we ſee them or not.

CHAP. III.

Of the addition and ſubſtraction of rectangles and ſquares.

WHEN the learner has made himſelf maſter of the demonſtrations contained in this ſecond book; it will be proper to turn his thoughts to the uſe which may be made of the properties of theſe rectangles and ſquares which he has been conſidering: and as almoſt the ſimpleſt uſe, which can be made of magnitudes, is adding them to one another or taking them from one another; it is very manifeſt that theſe properties will be in a manner uſeleſs, until we have acquired a readineſs in adding them together, and taking them from one another: and this indeed is one of the moſt intricate and extenſive principles in geometry. Much of this buſineſs may, and indeed ought to be learnt from this very book; for whoever ſhould treaſure up in his memory whatever is contained in this book, it would be but uſeleſs lumber unleſs conſidered under this particular point of view. And ſo much for the neceſſity of this practice; but a few examples will beſt explain my meaning.

Suppoſe in the ſecond propoſition you were required to add the rectangle contained by AB and BC to the rectangle contained by AB and AC; theſe taken together make the ſquare of AB: again if from the ſquare of AB you take away the rectangle contained by AB and BC; the remainder is the rectangle contained by AB and AC; all this is obvious from the inſpection of the figure; but as theſe conſequences may be wanted when the figure is not at hand for inſpection, the reader ought to be able to derive them readily from any ſingle ſtraight line cut into two ſegments. But again in the third propoſition if you add the rectangle contained by AC and CB to the ſquare of CB it makes the rectangle contained by AB and BC; or if you add the rectangle contained by AC and CB to the ſquare of AC it makes the rectangle contained by AB and AC: And farther if from the rectangle contained by AB and BC you take away the ſquare of BC there remains the rectangle contained by AC and CB.

In the fourth propoſition if you take the ſquares of AC and CB from the ſquare of AB there is left the rectangle contained by AC and CB taken twice: or if the ſquare of AC be added to the rectangle contained by AC and CB taken twice; their ſum will be equal to the difference between the ſquares of AB and BC; and this is alſo obvious from the inſpection of the figure.

The fifth propoſition, in which is compared the rectangles made by the equal ſegments of a line with thoſe made by the unequal ſegments, furniſhes frequent opportunities for this kind of practice; thus if from the ſquare of half the line BC you take away the ſquare of CD the remainder is the the rectangle contained by AD and DB; and this ſquare of CD added to the rectangle contained by AD and DB makes up the ſquare of half the line. And in the ſixth propoſition if from the ſquare of CD you take the ſquare of CB the remainder is the rectangle contained by AD and DB. One might juſt obſerve upon the fifth; that the rectangle contained by AD and DB and the ſquare of BC are bounded by the ſame extent of line; but that the ſpaces incloſed differ by the ſquare of CD.

It would be tedious to be more particular upon this ſubject; I ſhall therefore conclude with obſerving that ſquares are generally added together or ſubſtracted from one another by the forty ſeventh propoſition of the firſt book; and that they are doubled by the aſſiſtance of a right angled iſoſceles triangle as in the ninth and tenth propoſitions; and may be halved upon the ſame principles.

CHAP. IV.

In which is ſhewn the abſurdity of applying numbers to illuſtrate the propoſitions in this book.

IF you draw two indefinite ſtraight lines at right angles to each other, and cut off from one, ſeven equal parts, beginning at the angular point, and from the other four of the ſame equal parts, and compleat the right angled parallelogram; then, through each of the diviſions, drawing lines parallel to the ſides of the rectangle; the whole ſurface is divided into ſquares all equal and twenty eight in number, which is the product of four multiplied by ſeven. And becauſe this parallelogram is equal to any other upon the ſame or an equal baſe and between the ſame parallel lines; and a triangle in the ſame circumſtances, the half of it: therefore this other parallelogram, though not diviſible into ſquares, is ſaid to contain twenty eight ſuch ſquares; and the triangle fourteen of the ſame. And as any rectilineal figure is diviſible into triangles, upon theſe principles

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principles any rectilineal figure may have its contents expreſt in ſquare meaſure, inches feet or yards according to the meaſure by which you ſuppoſe your lines to be divided.

Now this way of expreſſing the ſurface or contents of a right angled parallelogram in ſquare meaſure by the product of the two ſides about one of the right angles, has introduced a very abſurd *practice* upon this ſecond book, which ſome facitious gentlemen have been pleaſed to ſtile illuſtrations.

But we ſhall be beſt able to decide how far this method of *operation* can be called an illuſtration by conſidering the uſe for which it is intended. Now the common affairs of life require, that we ſhould pay a particular attention to ſmall and large portions of extenſion within certain limits; which made it neceſſary to aſſume particular parts of extenſion as a common meaſure; and for this purpoſe at firſt certain parts of the human body ſeemed to have been uſed as is obvious from the names of meaſures; an inch, a foot, a ſpan &c. and very probably he who obſerved that the feet of different men were unequal in length, and thus proved the neceſſity of having ſome fixt ſtandard meaſure, ſtood pretty high in the opinion of his cotemporaries for ingenuity. Suppoſe them to have made the ſtandard meaſure which we call a foot; for ſome purpoſes this would be too large and for others too ſmall. The diſtance of one place from another expreſt in feet could bring no diſtinct idea to the mind; it would be an improvement to take three feet and make an unit of that, under the different name of a yard, but ſtill more to take one thouſand ſeven hundred and ſixty yards and call it a mile. And by ſuch contrivances our perceptions of diſtance might be made tollerably accurate and ſuited to our circumſtances. Surfaces are meaſured upon the ſame plan, and expreſt in ſquare feet, yards or miles upon the principles explained above: and to the great comfort of the unthinking part of mankind all ſurfaces may be compared in a manner ſufficiently accurate for the purpoſes of common life, not only without our ever attending to a ſurface; but what is more without our having any notion of extenſion. For our attention is turned to no particular kind of magnitude, as we are only to conſider whether one number be greater than another; and here it is not neceſſary to form your judgement by the nature of numbers, but by a particular method of notation which you have been taught.

It would be ſtrange enough to put one upon meaſuring ſuch figures as Euclid treats of in this ſecond book, in order to get at thoſe properties which he ſays belong to them; but it would be the greateſt abſurdity to call this an illuſtration of Euclid's demonſtrations; as the two methods are founded upon principles, in a certain ſenſe directly contrary to each other: theſe menſurations drawing the reader's attention from Euclid's plan to fix it upon ſomething elſe; or, more properly ſpeaking, upon nothing. Neither is the concluſion ſcientifically accurate; for the lines taken at a venture cannot be divided according to any meaſure; particularly if the line be cut as is required in the eleventh propoſition, it is impoſſible to meaſure it and its parts. But to reſt this matter entirely upon the moſt material objection, viz. that it is directly contrary to Euclid's ideas. He teaches how to turn any rectilineal figure into a ſquare in the laſt propoſition of this book. Which ſhews that his plan is to make every ſtep we take the object of the underſtanding, and therefore does not preſent a multitude of things to the mind which it is impoſſible for it to comprehend, but two diſtinct things for its contemplation: he does not expreſs irregular figures by a multitude of ſmall ſquares of any particular name; but reduces any two, which he may have occaſion to compare, to two ſquares; nor has his reaſoning a reference to any particular meaſure. In ſhort if my intention was to purchaſe figures, I might truſt to this practice; but if I meant to reaſon about their properties it muſt be entirely laid aſide.

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DISSERTATION V.

THE triangle and rectangle two of the great inſtruments of geometrical inveſtigation, being conſidered, Euclid proceeds to the third, which is of no leſs importance than either of the other two. And is here in its proper place, becauſe the properties of the circle are derived, partly from the triangle, and partly from the rectangle.

In the ſecond book the propoſitions are arranged according to their ſimplicity; and not as in the firſt book according to the dependence which the propoſitions have upon each other: becauſe the tenth might be read firſt, for any connexion which it has with the firſt nine propoſitions; the four laſt indeed are connected with ſome of the preceeding propoſitions, but none of the others. In this third book the arrangement is made partly according to the ſimplicity of the properties and partly according to the connexion which they have with one another; and the explanation of this arrangement with ſome remarks upon the method of demonſtration ſhall be the ſubject of this diſſertation.

CHAP. I.

Containing remarks upon the arrangement of the propoſitions.

IN the two firſt books our author conſiders the circle only as a mechanical inſtrument, and the uſe made of it, reſts entirely upon the third poſtulate; and unleſs it were introduced upon a different footing, it could hardly be reckoned a geometrical figure. In ſhort in the firſt two books it is only a pair of compaſſes with an indefinite extent and perfectly accurate. But here in this book it puts on a very different character, and is a geometrical figure bounded by one line called the circumference; and it is alſo added that there is a certain point within the figure from which all the lines drawn to this circumference are equal; but this point is to be found, before we can expect to make much of the properties of the circle. The firſt propoſition teaches us how to find a ſtraight line paſſing through this point; and not ſimply a ſtraight line, but a finite ſtraight line, the cutting which in halves will find the center. This being found it ſeems both natural and neceſſary to conſider whether the circumference of a circle be totally and eſſentially different from a ſtraight line; which it is proved to be in the ſecond, by ſhewing that every ſtraight line joining any two points in the circumference falls entirely within the circle, and can neither apply itſelf to any part of the circumference, nor even meet it except in the two aſſumed points; from which it follows that a ſtraight line cannot cut the circumference of a circle in more points than two.

But to return to the center; the inveſtigation of this point would ſhew that the center will always be found in any ſtraight line cutting another, terminated by the circle, in halves and at right angles: this diſcovery brings us directly to the third propoſition; for it would not be difficult to imagine that any ſtraight line drawn from the center to the point in which any line was cut in halves, would fall in with this perpendicular, and the contrary; and this would lead to the demonſtration of the fourth propoſition, which muſt have been ſuggeſted already by the property of the center, *that all lines paſſing through it and terminated by the circle are cut in halves there*; becauſe it would be proper to demonſtrate that no other point has this property, and it is here ſhewn that not even two lines terminated by the circle can be cut in halves in any other point. But not to be tedious, the meditating upon the center would very naturally lead to the diſcovery of the firſt fifteen propoſitions; and it is hardly to be imagined that a geometrician would drop the contemplation of the ſecond propoſition before he had diſcovered in what manner a ſtraight line drawn through any point in the circumference of the circle would meet its circumference, which would lead to the conſideration of the ſixteenth, ſeventeenth, eighteenth and nineteenth propoſitions. And it would be very proper for the ſtudent to examine the firſt nineteen propoſitions again and again, before he proceed any farther in the book.

CHAP. II.

The ſame ſubject continued.

THE firſt and ſecond propoſitions might lead in a very eaſy and natural manner to the diſcovery of the firſt nineteen propoſitions: but the idea of ſimilar ſegments is by no means an obvious one; for the common notion is to the laſt degree vague and undeterminate: ſo that ever ſince I could pretend to form any opinion upon this ſubject, I have always regarded this as one of the moſt maſterly parts of geometry, of the moſt ſubtile invention, and as having the moſt extenſive and unexpected conſequences; and yet when it is once conſidered that the angle at the center is double of the angle at the circumference, it all follows in a ſimple and ſeemingly obvious manner; for when it is once proved that all angles in the ſame ſegment are equal; if they ſhould once begin to conjecture that theſe angles did not depend upon any particular circle, but were fixt by the ſegment of the circle which they were conſidering, they would then arrive at their definition of ſimilar ſegments, and the whole buſineſs was done: and ſegments of circles could be compared with each other not in a vague manner but accurately and with the ſame preciſion as rectilineal figures; and the equality of the circumferences of equal circles could be aſcertained in a manner full as ſatisfactory as the baſes of the two triangles are proved to be equal in the fourth propoſition of the firſt book; and thus we could take one circumference double or treble of another, with the ſame eaſe that one ſtraight line may be made double or treble of another. It is not eaſy to ſay poſitively, what led them to their definition of ſimilar ſegments, but the moſt natural ſtep to it ſeems to be the proving that the angles in any ſemicircle are right angles, and that there ſeemed to be no other circumſtance about a ſegment fixt except the angles which it contained. Thoſe who judge of the demonſtration of a propoſition by the number of ſteps which it contains are apt to paſs over the demonſtrations of ſome of theſe propoſitions, as of little difficulty and leſs importance; but to the judicious they will appear, as they really are, of the firſt importance in this ſcience. The learner who would be properly acquainted with this ſubject ſhould conſider the propoſitions from the nineteenth to the thirty fifth, as all tending to the ſame purpoſe, and after giving them a very particular ſeparate examination, carefully trace their connexion with each other.

In the thirty fifth propoſition it is ſaid that the rectangles contained by the ſegments of lines interſecting one another within a circle are always equal; and the thirty ſixth is, that the ſquare of any tangent is equal to the rectangle contained by the whole cutting line, drawn from the ſame point, and the ſegment of it without the circle. From which it follows that if any number of ſtraight lines be drawn from a point without a circle, cutting the circle, the rectangles contained by the whole line and the part without the circle are all equal, which is the ſame property as that delivered in the thirty fifth, when the lines interſect one another within the circle.

To conclude, the reader will perceive upon a careful examination of this book, that there are three great ſources from which the properties of the circle are derived, namely the center; the ſimilar ſegments; and the equal rectangles made by the ſegments of lines terminated by it. We may ſay indeed that they are all ultimately derived from the center; but it appears to me that the ſtudent will find it very uſeful to conſider them under theſe three diſtinct heads.

The plan of the fourth book is ſo very obvious and regular that it ſeems ſufficient juſt to advertiſe the reader, that it is taken up in performing four problems, inſcribing the figure within the circle deſcribing the figure about the circle, inſcribing a circle within the figure and deſcribing a circle about the figure.

CHAP. III.

Containing ſome remarks on the demonſtrations.

EUCLID has a certain idea of demonſtration which he invariably adheres to; and although it be perfectly regular and accurate, yet it does not always take in every circumſtance which beginners require to be put in mind of: this I mention to avoid the imputation of vanity, leſt any additional hints to fit the demonſtration more to the diſpoſition of beginners might be looked upon as an attempt to ſupply ſome defect in the demonſtration: whereas they are intended only to remind the reader, that he may miſtake the meaning of the demonſtration, by truſting to the firſt view which is preſented to his imagination. And I here again repeat, what has been mentioned before, that he ſhould examine the ſimpleſt inſtances firſt, making them gradually more general until he arrive at ſuch a concluſion as the nature of the thing requires, obſerving what conſequences have been drawn from every part of the ſuppoſition, and whether any conſequence depends upon a particular poſition of the lines; and this is to be done not from any hopes of correcting the author, but only to get the better of his own prejudices; becauſe the author's taſk would be endleſs and indefinite, if he ſhould undertake to give ſuch demonſtrations as would remove all the prejudices which we are apt to take up upon this ſubject.

In the ſecond book, there is ſomething ſo determinate in the poſition of the lines, that any variation which the figures can undergo, either in ſhape or ſize, will hardly occaſion any miſconception in the meaning of the propoſition. But this is ſo far from being the caſe with regard to the *figure* which is the ſubject of this book, that by enlarging its ſize it may ſeem to have very different properties: for a circle deſcribed only with *a radius* equal to the ſemidiameter of the Earth, will differ ſo little from a ſtraight line in its circumference, according to vulgar comprehenſion, that perhaps for an hundred yards it would make a more accurate ſtraight line than was ever drawn by any inſtruments: and how are the contacts, interſections and other properties of ſuch circles to be determined but by reaſoning ſcientifically from the nature of the circle. But neither the neceſſity nor beauty of ſuch demonſtrations can be perceived by thoſe who confine their notions to ſuch circles as they can deſcribe with a pair of compaſſes.

I have indeed humoured this kind of prejudice in the deſcription of the figures for the ſecond and ſixteenth propoſitions, becauſe I never found a beginner who could enlarge his notions of theſe figures to make the poſition which the demonſtration requires ſeem probable. The reader may therefore give theſe demonſtrations a turn more ſuitable to the figures by conſidering that when he has proved, in the ſecond propoſition that DB is longer than DE and that they are both drawn from the center, and as DB reaches only to the circumference, therefore the ſhorter line DE cannot extend ſo far. In the ſame manner, in the ſixteenth DA being proved to be longer than DG; and being both drawn from the center, and DA reaching only to the circumference the line DG cannot reach ſo far; therefore the point G is within the circle, and conſequently the line AF cuts the circle. Let no one imagine that I propoſe this as a demonſtration; I have only expreſt myſelf thus in compliance with vulgar prejudice; for the legitimate demonſtrations are thoſe given by Euclid.

In the eighth propoſition the reader might juſt obſerve, that DK may be proved to be ſhorter than DL; in the ſame manner that DF in the former part of the propoſition is proved to be leſs than DE; and it might be farther proper to compare this propoſition with the ſeventh; when it will appear that they both contain the ſame property of the circle, only the point is taken within the circle in the one and without it in the other; alſo it may be proper to remark that the point D muſt be in the plane of the circle, otherwiſe the line DM would not cut the circumference; and the ſame is to be obſerved when we are required to draw a tangent from a point without a circle: and that the tangent itſelf muſt be in the plane of the circle. But Euclid takes it uniformly for granted that all his lines are in the ſame plane until he comes to the eleventh book. In the ninth propoſition the reader might ſuppoſe the point E to be within the angle ADB and obſerve what difference it will make in the demonſtration; he cannot then prove that DC will be greater than DB and DB than DA; but he can prove that DB is greater than DC which is ſufficient for his purpoſe. It is poſſible to miſtake the meaning of the eleventh propoſition, as if it confined the contact of the circles to a ſingle point; but that is not its meaning; ſuppoſe any point of contact, the ſtraight line joining the centers will paſs through the very point which you take; for if the line FG was alſo to paſs through another point of contact, the abſurdity would be juſt as great, that GD ſhould be equal to

GH and at the ſame time greater, as that it ſhould be leſs than GH and at the ſame time greater. There is ſomething rather ſingular in the eighteenth propoſition, for it ſeems to me to be nothing but the corollary to the ſixteenth propoſition; becauſe a tangent there is a ſtraight line at right angles upon the extremity of the diameter; according this notion therefore the one proves that AE is perpendicular to AB; and the other that AB is perpendicular to AE which is certainly the ſame thing. Unleſs it ſhould be urged in favour of Euclid that the corollary does not ſay that every tangent is at right angles to the diameter paſſing through the point of contact; but only if it be at right angles it will be a tangent; but it appears to me that there is no room for this diſtinction in the preſent caſe; for when we ſay that AE is perpendicular to AB, it is not the converſe of that propoſition to ſay that AB is perpendicular to AE. It would be a proper exerciſe for the ſtudent in reading the ſeventeenth propoſition to prove that two tangents might be drawn from any point without a circle, and that they are equal, though this is not according to Euclid's plan.

I come now to the twentieth propoſition, which ſhould be examined with the greateſt care, by deſcribing figures for every diſtinct caſe, and varying the poſition of the angular point through the whole circumference BADC: after this is done it will be proper for the ſtudent to make an obtuſe angle at the circumference, which will give him a particular kind of angle at the center, and it might admit of ſome diſpute whether it is to be reckoned an angle in Euclid's ſenſe of the word; however it may be proved to be double of the obtuſe angle at the circumference of the circle. As this is a very fundamental propoſition, the reader may juſt obſerve, (what I have recommended to him for a conſtant practice) the conſequences which are drawn from every part of the ſuppoſition; the angles are ſaid to be at the center and circumference; and to ſtand upon the ſame circumference. Now if either of theſe circumſtances be omitted, the triangles will not be iſoſceles &c. The next propoſition ſhould alſo be particularly examined in all its caſes; when the ſegment is greater than a ſemicircle, equal to a ſemicircle, and leſs than one; indeed if an angle increaſing until the lines take the ſame direction is to be conſidered as equal to two right angles, and if what an angle wants of four right angles is to be regarded as an angle, there is but one caſe of the propoſition; however if theſe do not make an angle, yet by drawing a line through the center they will be angles in Euclid's ſenſe of the term, and the demonſtration of the other two caſes will be obvious.

I ſhall conclude my remarks upon this book by deſiring the reader to take notice of a theorem which follows from the thirty third propoſition; viz. If two triangles have their baſes equal, and alſo the angles under which the baſes are extended; and if they be between the ſame parallel lines; the two triangles will be equal in every reſpect. This is mentioned to ſhew that the joining the properties of the circle to thoſe of the triangle will enlarge our notions very much; becauſe we never could have diſcovered this by the triangle alone.

In the fifth propoſition of the fourth book, *Simſon* thinks that Euclid ſhould have proved that the perpendiculars to the ſides of the triangles will meet; and it ſeems to me equally neceſſary to ſhew that the tangents will meet in the third propoſition; indeed if they did not the angle AKB would ceaſe to be an angle, and AKB would be a ſtraight, but it is equal to the outward angle of a triangle therefore &c. Or in both theſe inſtances it may be demonſtrated as *Clavius* has done by joining AB in the one; and DE in the other; which brings them to the eleventh common notion.

But although I recommend it to every ſtudent to examine all poſſible ſuppoſitions, yet for reaſons which have been mentioned already ſuch omiſſions ſeem to me very conſiſtent with Euclid's plan of demonſtration; for he ſuppoſes his reader attentive; and therefore gives him only, what he judged to be the neceſſary aſſiſtance.

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THE

ELEMENTS

OF

EUCLID.

BOOK I.

DEFINITIONS.

1. A POINT is that of which no part *can be taken*. 2. But A LINE *has* length without breadth. 3. Alſo extremities of a line *are* points. 4. Any *line* which lies evenly between the points in itſelf is A STRAIGHT LINE. 5. And that is A SURFACE which has length and breadth only. 6. Alſo extremities of a ſurface, *are* lines. 7. Any ſurface which lies evenly between the ſtraight lines in itſelf is A PLANE. 8. But A PLANE ANGLE is the inclination of lines to one another; *that is* of two lines in a plane, meeting each other and not lying in a ſtraight line. 9. Alſo it is called A RECTILINEAL ANGLE, when the lines containing the angle are ſtraight. 10. But when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another; each of the equal angles is A RIGHT ANGLE: and the ſtanding ſtraight line is called A PERPENDICULAR *to that* on

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Book I. which it ſtands. 11. AN OBTUSE ANGLE is that which is greater than a right angle. 12. But AN ACUTE *angle is* that which is leſs than a right angle. 13. That which is an extremity of any thing is *called* A TERM. 14. The *ſpace* bounded by one or more terms is called a FIGURE. 15. A CIRCLE is a plane figure bounded by one line, which is called a CIRCUMFERENCE: upon which all the ſtraight lines falling, from ONE POINT of thoſe lying within the figure, are equal to one another. 16. And the point is called the CENTER of the circle. 17. But a DIAMETER of a circle is any right line drawn through the center, and terminated both ways by the circumference of the circle: which alſo divides the circle into two equal parts. 18. Alſo A SEMI-CIRCLE is the figure bounded by a diameter, and the circumference of the circle cut off by it. 19. A SEGMENT of a circle is *the figure* bounded by any ſtraight line and the circumference of a circle. 20. The *figures* bounded by ſtraight lines are *called* RECTILINEAL FIGURES. 21. *Of theſe,* ſuch as are *bounded* by three, *are called* TRILATERAL. 22. As thoſe by four QUADRILATERAL. 23. But thoſe bounded by more than four ſtraight lines *are called* MULTILATERAL. 24. Again of trilateral figures, AN EQUILATERAL TRIANGLE particularly is that which has three equal ſides. 25. But an ISOSCELES, that which has only the two ſides equal. 26. And a SCALENE, that which has the three ſides unequal. 27. But moreover of trilateral figures, that again is a RIGHT-ANGLED TRIANGLE, which has a right angle. 28. And an OBTUSE ANGLED *triangle,* that which has an obtuſe angle. 29. Alſo that which has three acute angles, an ACUTE-ANGLED *triangle.* 30. But of quadrilateral figures, that is a SQUARE, which is equilateral and rectangular. 31. And an OBLONG, that which though rectangular is not equilateral. 32. And a RHOMBUS, though equilateral, is not rectangular. 33. But that which is neither equilateral, nor rectangular, having only its oppoſite ſides and angles equal to one another, *is* a RHOMBOIDES. 34. And *all other* quadrilateral figures except theſe may be called TRAPEZIUMS. 35. Any ſtraight lines, which are in the ſame plane, and being produced indefinitely towards both ſides meet each other on neither, are PARALLELS.

POS-

POSTULATES.

1. Let it be taken for granted, that a ſtraight line may be drawn from any one point to any other point. 2. And that a finite ſtraight line may be produced in a ſtraight line continually. 3. Alſo that a circle may be deſcribed with any center and *at any* diſtance.

COMMON NOTIONS.

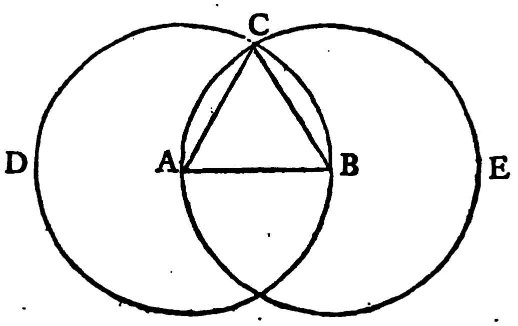
1. Magnitudes, which are equal to the ſame magnitude, are equal to one another. 2. And if equal ones be added to equal ones, the wholes are equal. 3. And if from equals, equals be taken away, the remainders are equal. 4. And if to unequals, equals be added, the wholes are unequal. 5. Alſo if from unequals, equals be taken away, the remainders are unequal. 6. And the doubles of the ſame are equal to one another. 7. Alſo the halves of the ſame are equal to one another. 8. Magnitudes which fit each other exactly are equal to one another. 9. Alſo the whole is greater than its part. 10. And all right angles are equal to one another. 11. And if a ſtraight line meeting two ſtraight lines, make thoſe angles, which are inward and upon the ſame ſide of it, leſs than two right angles, the two ſtraight lines being produced indefinitely will meet each other on the ſide where the angles are leſs than two right angles. 12. Alſo two ſtraight lines do not incloſe a ſpace.

PROPOSITION I.

Upon a given finite ſtraight line to deſcribe an equilateral triangle.

Let AB be the given finite ſtraight line: it is required to deſcribe an equilateral triangle upon the ſtraight line AB.

With the center A and at the diſtance AB (by poſt. 3.) let the circle BCD be deſcribed: and again, with the center B, but at the diſtance BA (by poſt. 3.) let the circle ACE be deſcribed: and from the point C in which the circles cut one another (by poſt. 1.) let the ſtraight lines CA, CB, be drawn to the points A and B.



Since therefore the point A is the center of the circle BCD (by Def. 15.) AC is equal to AB: again, becauſe the point B is the center of the circle ACE (by Def. 15.) BC is equal to BA. But it has been alſo demonſtrated that CA is equal to AB; each therefore of the *ſtraight lines* CA, CB is equal to AB; but (by com. not. 1.) magnitudes which are equal to the ſame magnitude are equal to one another, and the *ſtraight line* CA is therefore equal to the *ſtraight line* CB; wherefore the three ſtraight lines CA, AB, BC are equal to one another.

The triangle ABC is therefore (by Def. 24.) equilateral, and is deſcribed upon the given finite ſtraight line AB. Which was to be done.

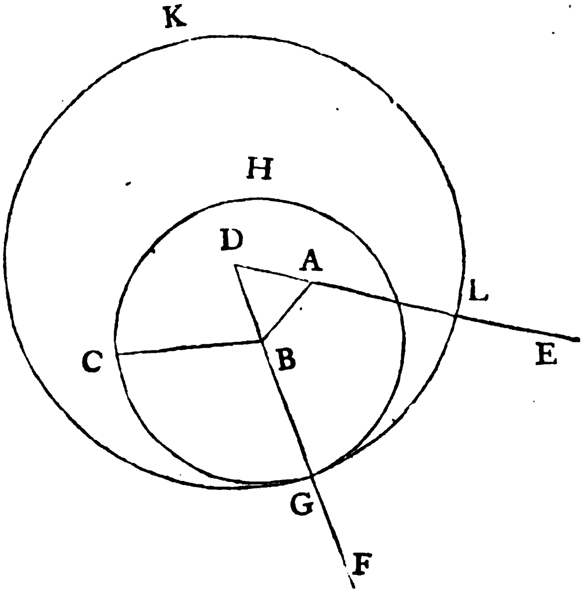
PROP. II.

At a given point, to place a ſtraight line, equal to a given ſtraight line.

Let the given point be the *point* A, and the given ſtraight line the *ſtraight line* BC: it is required at the point A to place a ſtraight line equal to the ſtraight line BC.

For let the ſtraight line AB be drawn (by poſt. 1.) from the point A to the point B: and upon it let the equilateral triangle DAB be deſcribed (by prop. 1.) and let the ſtraight lines AE, BF be produced (by poſt. 2.) in ſtraight lines with DA and DB; and then with the point B as a center and at the diſtance BC, let (by poſt. 3.) the circle CGH be deſcribed: and again, with the point D as a center, and at the diſtance DG let (by poſt. 3.) the circle GKL be deſcribed.

Since



Since therefore the point B is the center of the circle CGH the *ſtraight line* BC is (by def. 15.) equal to BG: and again, becauſe the point D is the center of the circle GKL the *ſtraight line* DL is equal to the *ſtraight line* DG (by Def. 15.) the parts of which viz. the *ſtraight line* DA is equal to the *ſtraigt line* DB, therefore the *ſtraight line* AL the remainder is equal (by com. not. 3.) to the *ſtraight line* BG the remainder, but the *ſtraight line* BC has been ſhewn to be equal to the *ſtraight line* BG: each therefore of the *ſtraight lines* AL, BC is equal to the *ſtraight line* BG; but magnitudes equal to the ſame, are alſo equal to one another: and therefore the *ſtraight line* AL is equal to the *ſtraight line* BC.

Wherefore at the given point A, the ſtraight line AL is placed equal to the given ſtraight line BC. Which was to be done.

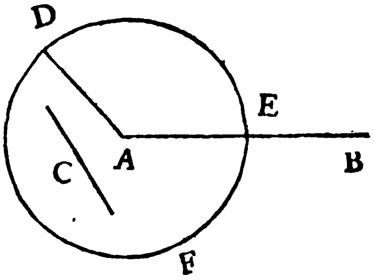
PROP. III.

Two unequal ſtraight lines being given, to cut off a part from the greater equal to the leſs.

Let the *ſtraight lines* AB and C be the two unequal ſtraight lines given: it is required from the greater the *ſtraight line* AB to cut off a ſtraight line equal to the ſtraight line C the leſs.

Place (by prop. 2.) at the point A a ſtraight line AD equal to the *ſtraight line* C; and then with the *point* A for a center, but at the diſtance AD, let the circle DEF be deſcribed.

And



And ſince the point A is the center of the circle DEF, the ſtraight line AE is equal to the ſtraight line AD (by def. 15.) but the *ſtraight line* C is alſo equal to the *ſtraight line* AD: each therefore of the *ſtraight lines* AE and C is equal to the *ſtraight line* AD: wherefore alſo the *ſtraight line* AE is equal (by com. not. 1.) to the *ſtraight line* C.

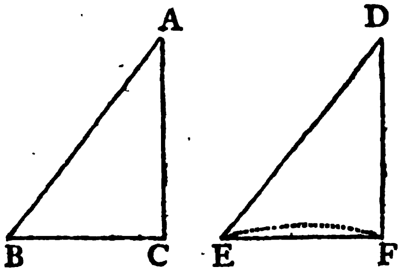
Wherefore two unequal ſtraight lines AB and C being given, from the *ſtraight line* AB the greater, the *ſtraight line* AE has been cut off equal to the *ſtraight line* C the leſs. Which was to be done.

PROP. IV.

If two triangles have the two ſides equal to the two ſides, each to each, and have the angle equal to the angle, the *angle* contained by the equal ſtraight lines: they ſhall alſo have the baſe equal to the baſe, and the triangle ſhall be equal to the triangle; and the other angles under which the *two* equal ſides are extended, ſhall be equal to the other angles, each to each.

Let the *triangles* ABC, DEF be two triangles having the two ſides the *ſtraight lines* AB, AC equal to the two ſides the *ſtraight lines* DE, DF each to each, the one, the *ſtraight line* AB to the *ſtraight line* DE; the other the *ſtraight line* AC to the *ſtraight line* DF; and an angle, that contained by BA, AC *or BAC* equal to that *contained* by ED, DF *or EDF:* I ſay, that alſo the *ſtraight line* BC a baſe is equal to the *ſtraight line* EF a baſe: and the triangle ABC ſhall be equal to the triangle DEF, and the other angles, under which the *two* equal ſides are extended, ſhall be equal to the other angles each to each; the one, the angle *contained* by AB, BC *or ABC* to that *contained* by DE, EF *or DEF*; the other, that *contained* by AC, CB *or ACB* to that *contained* by DF, FE *or* DFE.

For



For the triangle ABC being applied to the triangle DEF; and the point A in particular being put upon the point D, but the ſtraight line AB upon the upon the *ſtraight line* DE, alſo the *point* B will apply itſelf to the *point* E, becauſe the *ſtraight line* AB is equal (by ſuppoſition) to the *ſtraight line* DE: but the *ſtraight line* AB having applied itſelf to the *ſtraight line* DE, the ſtraight line AC will alſo apply itſelf to the *ſtraight line* DF, becauſe the angle *contained* by BA, AC *or BAC* is equal to that *contained* by ED, DF *or EDF* (by ſupp.): wherefore alſo, becauſe again (by ſupp.) the *ſtraight* line AC is equal to the *ſtraight line* DF, the point C will apply itſelf to the point F: certainly alſo the *point* B has applied itſelf to the *point* E, wherefore the *ſtraight line* BC a baſe, will apply itſelf to the *ſtraight line* EF a baſe: for if, the one, the *point* B having applied itſelf to the *point* E, and the other, the *point* C to the *point* F, the *ſtraight line* BC a baſe will not apply itſelf to the *ſtraight line* EF, two ſtraight lines will incloſe a ſpace, which (by com. not. 12.) is impoſſible: therefore the *ſtraight line* BC a baſe will apply itſelf to the *ſtraight line* EF, and (by com. not. 8.) will be equal to it: wherefore alſo the whole, the triangle ABC will apply itſelf to the whole, the triangle DEF, and (by com. not. 8.) will be equal to it: and the other angles will apply themſelves to the other angles, and (by com. not. 8.) will be equal to them; the one *contained* by AB, BC or ABC to that *contained* by DE, EF *or DEF*; the other *contained* by AC, CB *or ACB* to that *contained* by *DF, FE or DFE*.

If therefore two triangles have the two ſides equal to the two ſides, each to each, and have the angle equal to the angle, the *angle* contained by the equal ſtraight lines: they ſhall alſo have the baſe equal to the baſe, and the triangle ſhall be equal to the triangle; and the other angles, under which the *two* equal ſides are extended, ſhall be equal to the other angles, each to each. Which was to be demonſtrated.

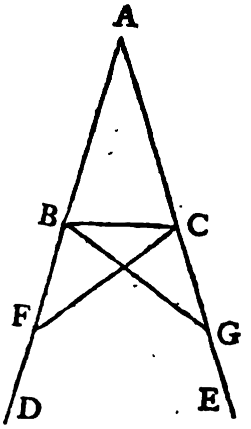
PROP.

PROP. V.

The angles at the baſe of iſoſceles triangles are equal to one another; and if the equal ſtraight lines be produced, the angles under the baſe ſhall be equal to one another.

Let the *triangle* ABC be an iſoſceles triangle, having the ſide AB equal to the ſide AC, and let the ſtraight lines BD, CE be produced in ſtraight lines to AB, AC; I ſay that the one, the angle *contained* by AB BC *or ABC* is equal to the *angle contained* by AC, CB *or ACB*; the other, *contained* by CB, BD *or CBD* to the *angle contained* by BC, CE *or BCE*.

Let any point which you may accidentally happen upon, as the *point* F be taken in the line BD, and let the *ſtraight line* AG be cut off (by prop. 3.) from the greater the *ſtraight line* AE, equal to the leſs the ſtraight line AF; and let the ſtraight lines FC, GB be drawn.



Since therefore the one *ſtraight line* AF is equal to the ſtraight line AG and the other AB to the *ſtraight line* AC, certainly the two FA, AC, are equal to the two, GA, AB, each to each, and they contain a common angle, the *angle contained* by FA, AG *or FAG:* therefore (by the 4th prop.) the ſtraight line CF a baſe is equal to the *ſtraight line* GB a baſe, and the triangle AFC will be equal to the triangle AGB, and the other angles will be equal to the other angles, each to each, under which the equal ſides are extended, the one *contained* by AC, CF or ACF to the *angle contained* by AB, BG *or ABG*, and the other *contained* by AF, FC *or AFC* to the *angle contained* by AG, GB *or AGB*.

And ſince a whole the *ſtraight line* AF is equal to a whole the *ſtraight line* AG, *parts* of which the *ſtraight line* AB is equal to the *ſtraight line* AC, wherefore (by com. not. 3.) the *ſtraight line* BF a remainder is equal to the *ſtraight line* CG a remainder: but the *ſtraight line* FC has been proved equal to the *ſtraight line* GB; certainly the two BF, FC are equal to the two CG, GB, each to each; and an angle the *angle contained* by BF, FC or BFC is equal to an angle, the *angle contained* by CG, GB or CGB, and the *ſtraight line* BC is a common baſe of them. And therefore (by the 4th prop.) the triangle BFC will be equal to the triangle CGB, and the other angles, under which the *two* equal ſides are extended, will be equal to the other angles, each to each; wherefore the one *contained* by FBC is equal to the *angle contained* by GCB; and the other *contained* by BCF to the angle contained by CGB. Becauſe therefore a whole the angle *contained* by ABG has been proved equal to a whole the angle *contained* by ACF, *parts* of which, the *angle contained* by CBG is equal to that *contained* by BCF, therefore the *angle contained* by ABC which remains is equal (by com. not. 3.) to that *contained* by ACB which remains, and they are at the baſe of the triangle ABC; but the *angle contained* by FBC has been alſo proved equal to that *contained* by GCB, and they are under the baſe.

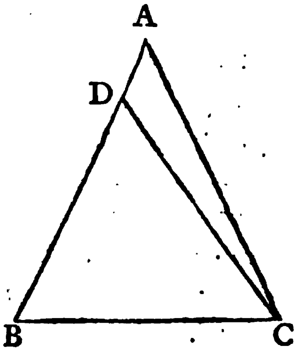
Wherefore the angles at the baſe of iſoſceles triangles are equal to one another; and if the equal ſtraight lines be produced, the angles under the baſe ſhall be equal to one another. Which was to be demonſtrated.

PROP. VI.

If the two angles of a triangle be equal to one another, alſo the ſides extended under the equal angles, ſhall be equal.

Let there be a triangle the *triangle* ABC, having the angle *contained* by ABC equal to the angle contained by ACB: I ſay alſo that a ſide, the *ſtraight line* AC, is equal to a ſide the ſtraight line AB.

For if the *ſtraight line* AC is unequal to the *ſtraight line* AB, the one of them is greater: let the ſtraight line AB be the greater; and (by prop. 3.) let the ſtraight line DB be cut off from the greater the *ſtraight line* AB equal to the leſs the *ſtraight line* AC, and let the *ſtraight line* DC be drawn.



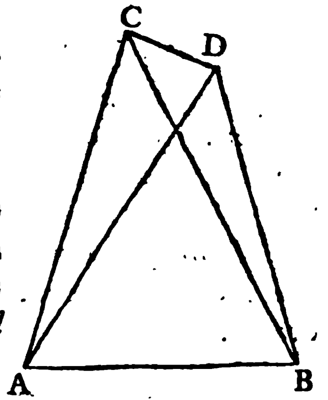
Since therefore the *ſtraight line* DB is equal to the *ſtraight line* AC, but the *ſtraight line* BC common; certainly the two DB, BC are equal to the two AC, CB, each to each; and an angle, the *angle contained* by DBC is equal to the angle *contained* by ACB (by ſupp.); wherefore (by prop. 4.) the *ſtraight line* DC a baſe is equal to the *ſtraight line* AB a baſe, and the triangle ABC will be equal to the triangle DCB, the greater to the leſs which is abſurd (by com. not. 9.); wherefore the *ſtraight line* AB is not unequal to the *ſtraight line* AC; therefore equal.

If therefore the two angles of a triangle be equal to one another, alſo the ſides, extended under the equal angles, ſhall be equal. Which was to be demonſtrated.

PROP. VII.

Upon the ſame ſtraight line, other two ſtraight lines, equal each to each to the ſame two ſtraight lines, will not join together at different points towards the ſame parts, if they have the ſame extremities as the firſt ſtraight lines.

For if it be poſſible, upon the ſame ſtraight line the *ſtraight line* AB, let other two ſtraight lines, the *ſtraight lines* AD, DB equal, each to each, to the ſame two ſtraight lines, the *ſtraight lines* AC, CB, be joined together at different points, the *points* C and D; the *points* C and D being towards the ſame parts; having the ſame extremities the *points* A, B as the firſt ſtraight lines; ſo that the one CA be equal to the *ſtraight line* DA, having the ſame extremity with it, the *point* A; and the other CB to the *right line* DB, having the ſame extremity with it, the *point* B: and let the ſtraight line CD be drawn.



Since therefore the *ſtraight line* AC is equal to the *ſtraight line* AD, alſo an angle the *angle contained* by ACD is equal (by prop. 5.) to the *angle contained* by ADC: wherefore the *angle contained* by ADC is greater than the *angle contained* by DCB; therefore the *angle contained* by CDB is is greater by much than the *angle contained* by DCB.

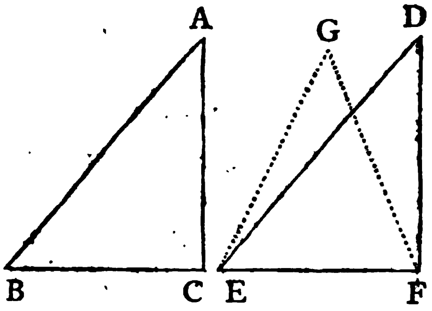
Since again the *ſtraight line* CB is equal to the *ſtraight line* DB (by ſupp.) alſo an angle the *angle contained* by CDB is equal (by prop. 5.) to the *angle contained* by DCB: but it has been ſhewn to be much greater than it, which is impoſſible.

Therefore upon the ſame ſtraight line, other two ſtraight lines equal, each to each, to the ſame two ſtraight lines will not join together at different points towards the ſame parts, if they have the ſame extremities as the firſt ſtraight lines. Which was to be demonſtrated.

PROP. VIII.

If two triangles have *not only* the two ſides equal to the two ſides, each to each, but have likewiſe the baſe equal to the baſe; they ſhall alſo have the angle equal to the angle, the *angle* contained by the equal ſtraight lines.

Let there be two triangles, the *triangles* ABC, DEF having the two ſides the *ſtraight lines* AB, AC equal to the two ſides the *ſtraight lines* DE, DF, each to each, the one AB to the *ſtraight line* DE, and the other AC to the *ſtraight line* DF; but let them have alſo a baſe the *ſtraight line* BC equal to a baſe the *ſtraight line* EF, I ſay alſo that an angle, the *angle contained* by BAC is equal to an angle, the *angle contained* by EDF.



For the triangle ABC being applied to the triangle DEF, and *not only* the point B being put upon the point E, but alſo the ſtraight line BC being put upon the *ſtraight line* EF, the point C alſo will apply *itſelf* to the *point* F, becauſe the *ſtraight line* BC is equal to the *ſtraight line* EF: certainly the *ſtraight line* BC having applied *itſelf* to the *ſtraight line* EF, alſo the *ſtraight lines* BA, AC will apply *themſelves* to the *ſtraight lines* ED, DF: for if a baſe as the *ſtraight line* BC ſhould apply *itſelf* to a baſe the *ſtraight line* EF, but the ſides BA, AC do not apply themſelves to the *ſtraight lines* ED, DF, but change their direction *from them* as the *ſtraight lines* EG, GF; upon

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the ſame ſtraight line, other two ſtraight lines equal, each to each, to the ſame two ſtraight lines will join together at different points towards the ſame parts, having the ſame extremities; but (by prop. 7.) they do not join together; and the ſides BA, AC, cannot therefore not apply *themſelves* to the *ſtraight lines* ED, DF, the baſe BC applying itſelf to the baſe EF, wherefore they do apply themſelves, ſo that alſo an angle the *angle contained* by BAC will apply *itſelf* to the angle *contained* by EDF, and ſhall be equal to it (by com. not. 8.)

If therefore the two triangles have the two ſides equal to the two ſides, each to each, and have the baſe equal to the baſe; they ſhall alſo have the angle equal to the angle, that which is contained by the equal ſtraight lines, Which was to be demonſtrated.

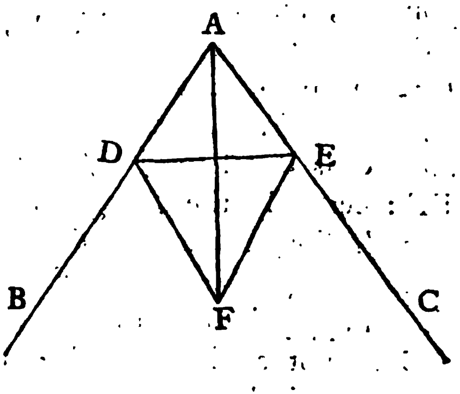
PROP. IX.

To cut a given rectilineal angle in halves.

Let the given rectilineal angle be, the angle contained by BAC: it is required to cut it in halves.

Let the point D be taken, any accidental *point* in the *ſtraight line* AB; and let the *ſtraight line* AE be cut off (by prop. 3.) from the *ſtraight line* AC equal to the *ſtraight line* AD, and let the ſtraight line DE be drawn (by poſt. 1.) and (by prop. 1.) let the equilateral triangle the *triangle* DEF be deſcribed upon the ſtraight line DE and (by poſt. 1.) draw the ſtraight line AF: I ſay that the angle contained by BAC is cut in halves by the ſtraight line AF.

For ſince the *ſtraight line* AD is equal to the *ſtraight line* AE, but the *ſtraight line* AF common, certainly the two, the *ſtraight lines* DA, AF are equal to the two EA, AF, each to each, and a baſe the the *ſtraight line* DF is equal to a a baſe the *ſtraight line* EF; therefore (by prop. 8.) an angle, the *angle contained* by DAF is equal to an angle, the *angle contained* by EAF.



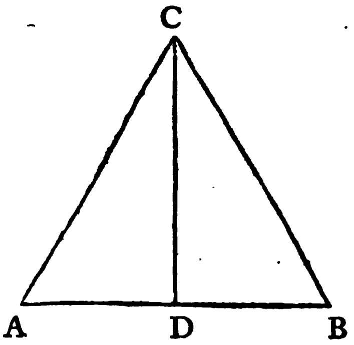
Wherefore the given rectilineal angle, the *angle contained* by BAC is cut in halves by the ſtraight line AF. Which was to be done.

PROP. II.

To cut a given finite ſtraight line in halves.

Let the *ſtraight line* AB be the given finite ſtraight line: it is required to cut the *ſtraight line* AB in halves.

Let an equilateral triangle the *triangle* ABC be deſcribed (by prop. 1) upon it, and let the angle *contained* by ACB be cut in halves (by prop. 9.) by the ſtraight line CD: I ſay that the ſtraight line AB is cut in halves in the point D.



For ſince the *ſtraight line* AC is equal (by conſtruction) to the *ſtraight line* CB, but the *ſtraight line* CD common, certainly the two AC, CD are equal to the two BC, CD, each to each, and an angle, the *angle contained* by ACD is equal (by conſt.) to the *angle contained* by BCD: therefore (by prop. 4.) a baſe the *ſtraight line* AD is equal to a baſe the *ſtraight line* BD.

Wherefore the given finite ſtraight line the *ſtraight line* AB is cut in halves in the *point* D. Which was to be done.

PROP. XI.

To draw a ſtraight line at right angles to any given ſtraight *line* from any point given in it.

Let the one, the given ſtraight line be the *ſtraight line* AB, and the other the given point in it the *point* C: it is required from the point C to draw a ſtraight line at right angles to the *ſtraight line* AB.

Let any point which you may accidentally happen upon, as the *point* D be taken in the *ſtraight line* AD; and let the ſtraight line CE be placed (by prop. 3.) equal to CD; and let an equilateral triangle the *triangle* DFE be deſcribed upon the *ſtraight line* DE, and let FC be drawn. I ſay that a ſtraight line the *ſtraight line* FC hath

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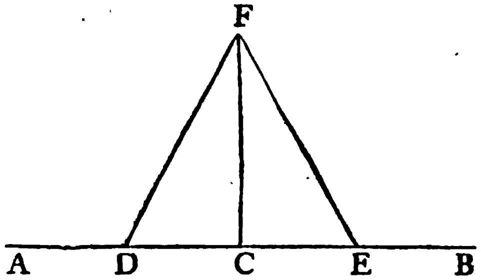
hath been drawn at right angles to the given ſtraight *line* the *ſtraight line* AB, from any point given in it the *point* C.



For ſince the ſtraight line CD is equal to the *ſtraight line* CE (by conſt.) and the ſtraight line CF common; certainly the two DC, CF are equal to the two EC, CF each to each; and a baſe the *ſtraight line* DF is equal (by conſt.) to a baſe the *ſtraight line* EF: wherefore (by prop. 8.) an angle the *angle contained* by DCF is equal to an angle the *angle contained* by ECF, and they are adjacent or *conſequent:* but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another; each of the equal angles is a right angle: therefore each of the *angles contained* by DCF, FCE is a right angle.

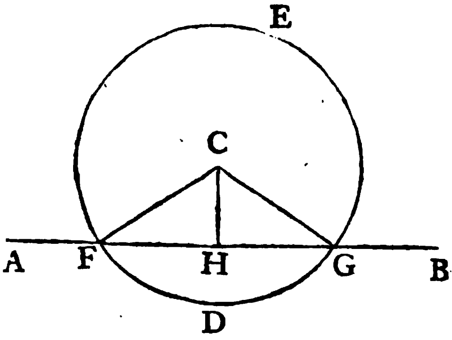
Wherefore a ſtraight line the *ſtraight line* FC hath been drawn at right angles to the given ſtraight *line* the ſtraight line AB, from any point given in it the *point* C. Which was to be done.

PROP. XII.

To draw a perpendicular ſtraight line upon any given indefinite ſtraight *line*, from any given point, which is not in it.

Let the one, the given indefinite ſtraight *line*, be the *ſtraight line* AB, and the other, the given point, which is not in it; *be* the *point* C; and it is required upon the given indefinite ſtraight *line*, the *ſtraight line* AB, from the given point, the *point* C, which is not in it, to draw a perpendicular ſtraight line.

For on the other ſide of the ſtraight line AB take any point which you may accidentally happen upon *as* the *point* D and with the *point* C indeed for a center, but at the diſtance CD let a circle be deſcribed, the *circle* FGE; and let the ſtraight line FG be cut in halves (by prop. 10.) in the *point* H, and let the *ſtraight lines* CF, CH, CG be drawn (by poſt. 1.): I ſay that upon the given indefinite ſtraight line, the *ſtraight line* AB, from the given point, the *point* C, which is not in it, a perpendicular hath been drawn, the *ſtraight line* CH. For



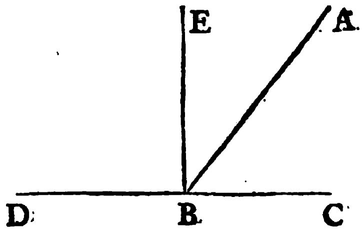
For ſince the ſtraight line FH is equal to the *ſtraight line* HG (by conſt.) and the ſtraight line HC common; certainly the two FH, HC are equal to the two GH, HC, each to each; and a baſe the ſtraight line CF is equal (by def. 15.) to a baſe the *ſtraight line* CG: wherefore (by prop. 8.) an angle, the *angle contained* by CHF is equal to an angle, the *angle contained* by GHC, and they are adjacent: but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another; each of the equal angles is a right angle: and the ſtanding ſtraight line is called a perpendicular *to that* on which it ſtands.

Therefore upon the given indefinite ſtraight line the *ſtraight line* AB, from the given point, the *point* C, which is not in it, a perpendicular hath been drawn, the *ſtraight line* CH. Which was to be done.

PROP. XIII.

Whenever a ſtraight line ſtanding upon a ſtraight line makes angles: it will make either two right angles, or *angles* equal to two right angles.

For let any ſtraight line, the ſtraight line AB, ſtanding upon a ſtraight line the *ſtraight line* CD, make angles, thoſe *contained* by CBA, ABD: I ſay that the angles *contained* by CBA, ABD are either two right angles, or equal to two right angles.



If indeed the *angle contained* by CBA be equal to the angle contained by ABD, they are (by def. 10.) two right angles: but if not, let the ſtraight line BE be drawn from the point B (by prop. 11.) at right angles to the *ſtraight line* CD: therefore the *angles contained* by CBE, EBD are two right angles: and ſince the *angle contained* by CBE is equal to two, the *angles contained* by CBA, ABE; let the angle *contained* by EBD, a common one be added: therefore (by com. not. 2.) the *angles contained* by CBE, EBD are equal to three, the *angles contained* by CBA, ABE, EBD. Again, becauſe the *angle contained* by DBA is equal to two, the the angles contained by DBE, EBA; let the angle contained by ABC a common one be added: therefore (by com. not. 2.) the angles, the *angles contained* by DBA, ABC are equal to three, the *angles contained* by DBE, EBA, ABC: but the *angles contained* by CBE, EBD have alſo been proved equal to the ſame three: but magnitudes which are equal to the ſame magnitude, are equal to one another: and the *angles contained* by CBE, EBD are therefore equal to the *angles contained* by DBA, ABC: but the *angles contained* by CBE, EBD are two right angles, and the *angles contained* by DBA, ABC are therefore equal to two right angles.

Whenever therefore a ſtraight line ſtanding upon a ſtraight line makes angles; it will make either two right angles, or *angles* equal to two right angles. Which was to be demonſtrated.

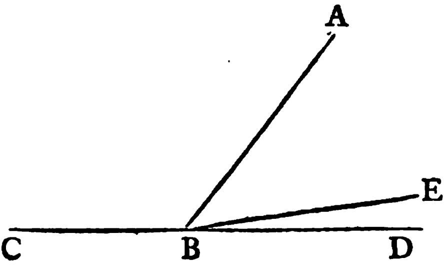
PROP. XIV.

If, with any ſtraight line, and at the *ſame* point in it, two ſtraight lines, not lying towards the ſame parts make the adjacent angles equal to two right angles, the ſtraight lines ſhall be in a ſtraight line with one another.

For with any ſtraight line, the *ſtraight line* AB, and at the *ſame* point in it, the *point* B, let two ſtraight lines, the *ſtraight lines* BC, BD not lying towards the ſame parts, make the adjacent angles, the *angles contained* by ABC, ABD equal to two right angles: I ſay that the ſtraight line BD is in a ſtraight line with the *ſtraight line* CB.

For if the *ſtraight line* BD is not in a ſtraight line with the *ſtraight line* CB; let the *ſtraight line* BE be in a *ſtraight line* with the ſtraight line CB.

Becauſe therefore a ſtraight line, the *ſtraight line* AB ſtands upon a ſtraight line, the *ſtraight line* CBE, therefore the angles *contained* by ABC, ABE are equal to two right angles: but the *angles contained* by ABC, ABD are alſo (by ſup.)



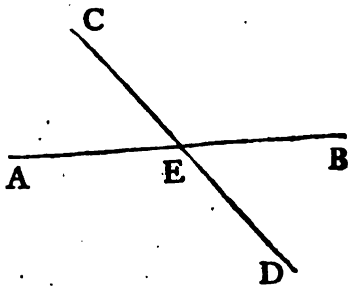
ſup.) equal to two right angles: therefore the *angles contained* by CBA, ABE are (by com. not. 1.) equal to the *angles contained* by CBA, ABD; let *what is* common, the angle *contained* by ABC be taken away, therefore what remains the *angle contained* by ABE is equal (by com. not. 3.) to what remains the angle contained by ABD, the leſs to the greater, which is impoſſible: therefore the ſtraight line BE is not in a ſtraight line with the *ſtraight line* BC: certainly in the ſame manner we ſhall ſhew that neither any other but the *ſtraight line* BD *is:* therefore the *ſtraight line* CB is in a ſtraight line with the *ſtraight line* BD.

Wherefore if, with any ſtraight line, and at the *ſame* point in it, two ſtraight lines, not lying towards the ſame parts, make the adjacent angles equal to two right angles, the ſtraight lines ſhall be in a ſtraight line with one another. Which was to be demonſtrated.

PROP. XV.

If two ſtraight lines cut one another they ſhall make the angles at the vertex equal to each other.

For let two ſtraight lines, the *ſtraight lines* AB, CD cut one another at the point E: I ſay that the one the *angle contained* by AEC is equal to the angle contained by DEB and the other *contained* by CEB to the *angle contained* by AED.



For ſince a ſtraight line, the *ſtraight line* AE ſtands upon a ſtraight line the *ſtraight line* CD, making angles, the *angles contained* by CEA, AED: therefore the angles *contained* by CEA, AED are equal to two right angles (by prop. 13.): again, becauſe a ſtraight line, the *ſtraight line* DE ſtands upon a ſtraight line, the *ſtraight line* AB, making angles the *angles contained* by AED, DEB: therefore the *angles contained* by AED, DEB are equal to two right angles: but alſo the *angles contained* by CEA, AED have been demonſtrated *to be* equal to two right angles: therefore (by com. not. 1.) the *angles contained* by CEA, AED are equal to the *angles contained* by AED, DEB: let the *angle contained* by AED

Book I. *which is* common be taken away, wherefore (by com. not. 3.) the *angle contained* by CEA which remains is equal to the *angle contained* by BED which remains: certainly in the ſame manner it will be ſhewn, that alſo the *angles contained* by CEB, DEA are equal.

If therefore two ſtraight lines cut one another they ſhall make the angles, at the vertex equal to each other. Which was to be demonſtrated.

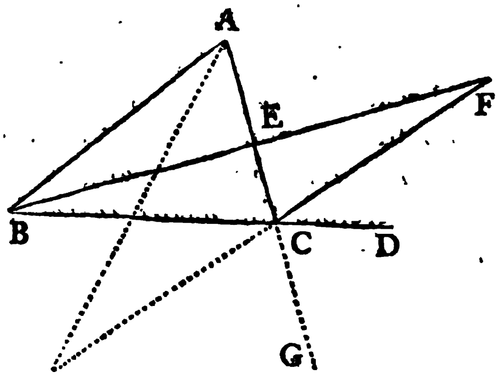
Corollary. Certainly it is manifeſt from this, that alſo whatever number of ſtraight lines cut one another, they will make the angles at the ſection equal to four right angles.

PROP. XVI.

One of the ſides of any triangle being produced, the outward angle is greater than either of thoſe *two which are* within and oppoſite.

Let there be a triangle, the *triangle* ABC, and let one ſide of it the ſtraight line BC be produced to the *point* D: I ſay that the angle without, the *angle contained* by ACD is greater than either of thoſe *two which are* within and oppoſite, the angles contained under CBA, BAC.

Let the ſtraight line AC be cut in halves in the point E (by prop. 10.); and the ſtraight line BE being drawn (by poſt. 1.) let it (by poſt. 2.) be produced to the *point* F, and let the ſtraight line EF be placed (by prop. 3.) equal to the *ſtraight line* BE and let the ſtraight line CF be drawn (by poſt. 1.), and let the ſtraight line AC be produced to G.



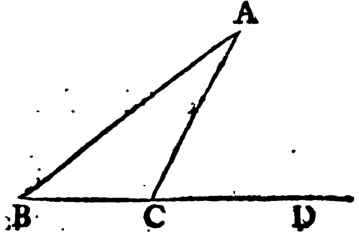
Since therefore the one *ſtraight line* AE is equal to the ſtraight line EC (by conſt.) and the other the *ſtraight line* BE to the *ſtraight line* EF; certainly the two AE, EB are equal to the two CE, EF, each to each, and an angle, the *angle contained* by AEB, is equal to an angle, the *angle* *contained* by FEC (by prop. 15.) for *they are* vertical: wherefore (by prop. 4.) a baſe the *ſtraight line* AB is equal to a baſe the ſtraight line CF and the triangle ABE is equal to the triangle FEC, and the remaining angles are equal to the remaining angles, under which the equal ſides are extended, each to each: wherefore the *angle contained* by BAE is equal to the *angle contained* by ECF: but the angle contained by ECD is greater (by com. not. 9.) than the *angle contained* by ECF: wherefore the *angle contained* by ACD is greater than the *angle contained* by BAE: in like manner alſo the ſtraight line BC having been cut in halves, the angle contained by BCG, that is (by prop. 15.) the *angle contained* by ACD, *is* alſo greater than the *angle contained* by ABC.

Wherefore one of the ſides of any triangle being produced, the outward angle is greater than either of thoſe *two which are* within and oppoſite. Which was to be demonſtrated.

PROP. XVII.

The two angles of any triangle are leſs than two right angles, being interchanged every way.

Let there be a triangle the *triangle* ABC; I ſay that the two angles of the triangle ABC are leſs than two right angles, being interchanged every way.



For let the *ſtraight line* BC be produced to the *point* D.

And ſince the angle contained by ACD is an outward angle of the triangle ABC, it is greater than the inward and oppoſite, the angle contained by ABC: let a common one be added, the *angle contained* by ACB: therefore the *angles contained* by ACD, ACB are greater (by com. not. 4.) than thoſe *contained* by ABC, BCA; but the *angles contained* by ACD, ACB are equal to two right angles; therefore thoſe *contained* ABC, BCA are leſs than two right angles. And in like manner we ſhall ſhew that the *angles contained* by BAC, ACB are leſs than two right angles, as alſo *thoſe contained* by CAB, ABC.

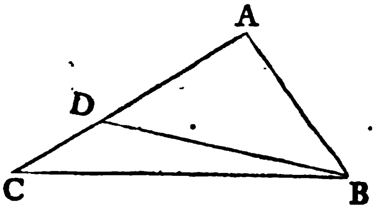
Wherefore the two angles of any triangle are leſs than two right angles; being interchanged every way.

PROP. XVIII.

The greater ſide of every triangle ſubtends the greater angle.

Let there be a triangle, the *triangle* ABC, having the ſide AC greater than the *ſide* AB, I ſay that alſo the angle contained by ABC is greater than the angle contained by BCA.

For becauſe the *ſtraight line* AC is greater than the *ſtraight line* AB let the *ſtraight line* AD be placed (by prop. 3,) equal to the ſtraight line AB, and let the *ſtraight line* BD be drawn.



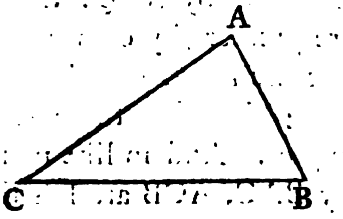
And ſince the *angle contained* by ADB is an outward angle of a triangle, the *triangle* BDC, it is greater (by prop. 16.) than the inward and oppoſite, the *angle contained* by DCB. But the *angle contained* by ADB is equal to the *angle* contained by ABD (by prop. 5.), becauſe the ſide AB is equal to the *ſide* AD: therefore alſo the *angle contained* by ABD is greater than the *angle* contained by ACB; wherefore the *angle contained* by ABC is much greater than the *angle contained* by ACB.

Therefore the greater ſide, of every triangle, ſubtends the greater angle. Which was to be demonſtrated.

PROP. XIX.

The greater ſide of every triangle is extended under the greater angle.

Let there be a triangle, the *triangle* ABC, having the angle *contained* by ABC greater than the *angle contained* by BCA; I ſay that alſo the ſide AC is greater than the ſide AB.



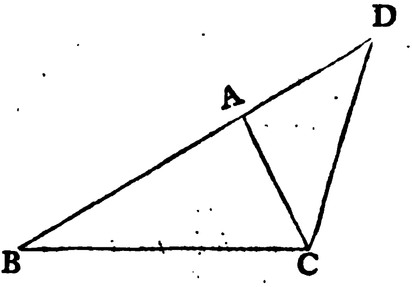
For if not: certainly the *ſtraight line* AC is equal to the *ſtraight line* AB, or leſs: but indeed the *ſtraight line* AC is not equal to the *ſtraight line* AB, for then alſo an angle, the *angle contained* by ABC would be equal (by prop. 5.) to the *angle contained* by ACB: but (by ſupp.) it is not: therefore the *ſtraight* *line* AC is not equal to the *ſtraight line* AB; neither is the *ſtraight line* AC leſs than the *ſtraight line* AB; for if it were leſs, then (by prop. 18.) an angle the *angle contained* by ABC *would be leſs* than the *angle contained* by ACB: but (by ſupp.) it is not: therefore the *ſtraight line* AC is not leſs than the *ſtraight line* AB; but it has been demonſtrated, that it is not equal *to it*; wherefore the *ſide* AC is greater than the ſide AB.

Wherefore the greater ſide of every triangle is extended under the greater angle. Which was to be demonſtrated.

PROP. XX.

The two ſides of every triangle are greater than the remaining ſide, being interchanged every way.

For let there be a triangle, the *triangle* ABC: I ſay that the two ſides of the triangle ABC are greater than the remaining *ſide*, being interchanged every way: viz. BA, AC *greater* than BC; and AB, BC *greater* than AC; laſtly BC, CA *greater* than AB.



For let the *ſtraight line* BA be produced to the point D, and let the ſtraight line DA be placed (by prop. 3.) equal to CA, and let the *ſtraight line* DC be drawn.

Since therefore the *ſtraight line* DA is equal to the ſtraight line AC, alſo (by prop. 5.) the *angle contained* by ADC is equal to the *angle contained* by ACD: but the angle *contained* by BCD is greater than the *angle contained* by ACD (by com. not. 9.): wherefore the *angle contained* by BCD is *alſo* greater than the *angle contained* by ADC: and becauſe the *triangle* DCB is a triangle having the angle contained by BCD greater than the *angle contained* by BDC, and (by prop. 19.) the greater ſide is extended under the greater angle, wherefore the *ſtraight line* DB is greater than the *ſtraight line* BC; but the ſtraight line BD is equal to the ſtraight lines AB, AC: therefore the ſtraight lines BA, AC are greater than the ſtraight line BC: certainly in the ſame manner we ſhall demonſtrate that alſo AB, BC are greater than AC; and BC, CA greater than AB.

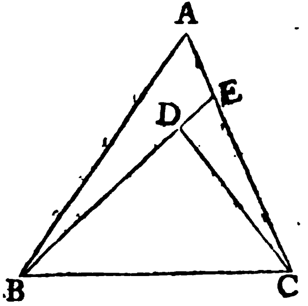
Therefore of every triangle the two ſides are greater than the remaining *ſide*, being interchanged every way. Which was to be demonſtrated.

PROP. XXI.

If two ſtraight lines be joined together within a triangle, upon one of its ſides from the extremities of it, the joined lines ſhall indeed be leſs than the two remaining ſides of the triangle, but will contain a greater angle.

For let two ſtraight lines, the *ſtraight lines* BD, DC be joined together within a triangle, the *triangle* ABC upon one of the ſides the *ſtraight line* BC *drawn* from the extremities the *points* B, C; I ſay that the *ſtraight lines* BD, DC are indeed leſs than the two remaining ſides of the triangle the *ſtraight lines* BA, AC; but contain a greater angle, the *angle contained* by BDC greater than the *angle contained* by BAC.

For let the *ſtraight line* BD be produced to the *point* E.



And becauſe the two ſides of any triangle are greater than the remaining *ſide*, therefore the two ſides of the triangle ABE, the ſtraight lines AB, AE are greater than the *ſtraight line* BE (by prop. 20.); let the ſtraight line EC a common *one* be added: therefore (by com. not. 4.) the *ſtraight lines* BA, AC are greater than the *ſtraight lines* BE, EC. Again, becauſe the two ſides of the triangle CED, the *ſtraight lines* CE, ED are greater than the ſtraight line CD, let the *ſtraight line* DB a common *one* be added; therefore (by com. not. 4.) the *ſtraight lines* CE, EB are greater than the *ſtraight lines* CD, DB: but the *ſtraight lines* BA, AC have been ſhewn *to be* greater than the *ſtraight lines* BE, EC; therefore BA, AC are greater by much than BD, DC.

Again, becauſe the outward angle of every triangle is greater than the inward and oppoſite (by prop. 16.): therefore the outward angle of the triangle CDE, the *angle contained* by BDC is greater than that *contained* by CED: For the ſame reaſon therefore alſo the outward angle of the triangle ABE, the *angle contained*

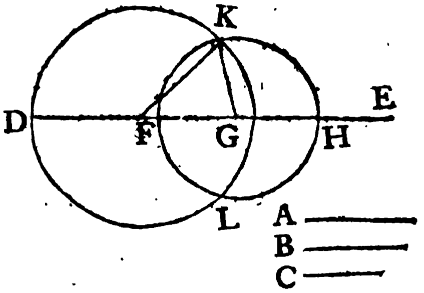
tained by CEB is greater than that *contained* by BAC: but the *angle contained* by BDC has been demonſtrated *to be* greater than that *contained* by CEB: therefore the *angle contained* by BDC is greater by much than that contained by BAC.

If therefore two ſtraight lines be joined together within a triangle, upon one of its ſides from the extremities of it, the joined lines ſhall indeed be leſs than the two remaining ſides of the triangle, but will contain a greater angle. Which was to be demonſtrated.

PROP. XXII.

To deſcribe a triangle of three ſtraight lines, which are equal to three given ſtraight lines: but it is neceſſary that two of them be greater than the remaining *one*; being interchanged every way.

Let the three given ſtraight lines be A, B, C; of which let the two be greater than the remaining *one*, being interchanged every way, that is A and B greater than C: A and C greater than B; as alſo B and C greater than A: it required to deſcribe a triangle of *ſtraight lines* equal to A, B, C.



Draw any ſtraight line DE terminated at the *point* D but indifinite towards the *point* E; and (by prop. 3.) let DF be made equal to A, and FG equal to B; but GH equal to C: and with the point F for a center, and at the diſtance FD let the circle DKL be deſcribed; and again with the *point* G for a center, and at the diſtance GH let the circle KLH be deſcribed; and draw KF, KG: I ſay that the triangle KFG hath been deſcribed of three ſtraight lines equal to A, B, C.

For becauſe the point F is the center of the circle DKL, FD is equal to FK: but FD is equal (by conſt.) to A, therefore FK is equal to A. Again becauſe the point G is the center of the circle LKH, (by def. 15.) GK is equal to GH; but GH is equal to C, therefore GK is equal to C: but FG is alſo equal to B; therefore the three ſtraight lines KF, FG, GK are equal to the three A, B, C.

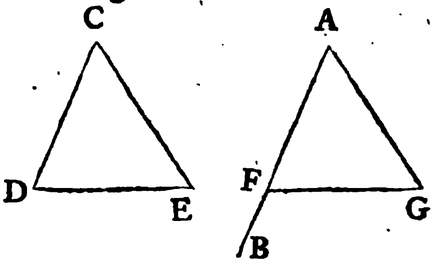
Therefore the triangle KFC is deſcribed of three ſtraight lines, which are equal to the three given ſtraight lines A, B, C. Which was to be done.

PROP. XXIII.

With a given ſtraight line, and at a point in it, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and the *point* A the point in it; and DCE the given rectilineal angle: but it is required with the given ſtraight line AB and at the point A in it to make a rectilineal angle equal to the given rectilineal angle DCE.

Take in each of the *lines* CD, CE any points whatever D and E, and join DE, and (by prop. 22.) deſcribe the triangle AGF of three ſtraight lines, which are equal to the three, CD, DE, CE; ſo that CD be equal to AF; CE to AG, and alſo DE to FG.



Since therefore the two DC, CE are equal to the two FA, AG, each to each, and the baſe DE equal to the baſe FG; therefore (by prop. 8.) the angle DCE is equal to the angle FAG.

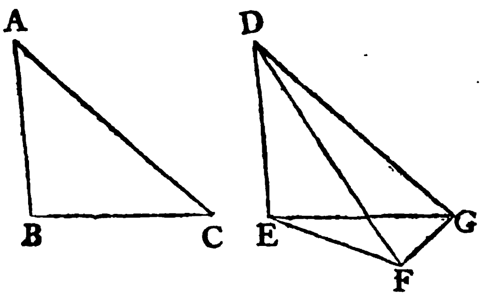
Wherefore with a given ſtraight line AB, and at a point in it the *point* A, a rectilineal angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

PROP. XXIV.

If two triangles have the two ſides equal to the two ſides, each to each, but have the angle greater than the angle, the *angle* contained by the equal ſtraight lines: alſo they will have the baſe greater than the baſe.

Let there be two triangles the *triangles* ABC, DEF having the two ſides AB, AC equal to the two ſides DE, DF, each to each; AB to DE, and AC to DF; but let an angle the *angle contained* by BAC be greater than the angle *contained* by EDF; I ſay that the baſe BC is greater than the baſe EF.

For



For becauſe the angle BAC is greater than the angle EDF; let there be made, with the ſtraight line DE and at the point D in it, the angle EDG equal to the angle BAC; and let DG be made equal (by prop. 3.) to either of the *lines* AC, DF; and let GE, GF be joined.

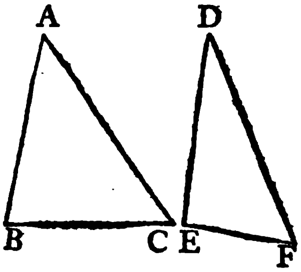
Since therefore AB is equal to DE and AB to DG; certainly the the two BA, AC are equal to the two ED, DG, each to each; and the angle BAC *is* equal (by conſt.) to EDG therefore the baſe BC is equal to the baſe EG. Again becauſe DG is equal to DF, the angle DFG is equal (by prop. 5.) to the angle DGF; therefore the *angle* DFG is greater than the *angle* EGF; therefore the *angle* EFG is greater by much than the *angle* EGF; and becauſe there is a triangle, the *triangle* EFG, having the angle EFG greater than the *angle* EGF; but (by prop. 19.) the greater ſide is extended under the greater angle: therefore the ſide EG is greater than EF: and EG *is* equal to BC (by part. 1. of this prop.); wherefore alſo BC is greater than EF.

Wherefore if two triangles have the two ſides equal to the two ſides, each to each, and have the angle greater than the angle, the *angle contained* by the equal ſtraight lines; they will alſo have the baſe greater than the baſe. Which was to be demonſtrated.

PROP. XXV.

If two triangles have the two ſides equal to the two ſides each to each, and have the baſe greater than the baſe; they will alſo have the angle greater than the angle, the *angle* contained by the equal ſtraight lines.

Let there be two triangles the *triangles* ABC, DEF having the two ſides BA, AC equal to the two ſides DE, DF, each to each, AB to DE and AC to DF; but let the baſe BC be greater than the baſe EF: I ſay alſo that an angle, the *angle* BAC is greater than an angle, the *angle* EDF.



For if not; certainly it is equal to it, or leſs. But certainly the angle BAC is not equal to EDF; for then (by prop. 4.) the baſe BC would be equal to the baſe EF: but it is not (by ſupp.); therefore the angle BAC is not equal to EDF: but neither indeed is it leſs; for if it were leſs, alſo the baſe BC *would be leſs* (by prop. 24.) than the baſe EF: but it is not (by ſupp.); therefore the angle BAC is not leſs than EDF: but it has been demonſtrated that neither is it equal: wherefore the angle BAC is greater than EDF.

Wherefore if two triangles have the two ſides equal to the two ſides each to each, and have the baſe greater than the baſe; they will alſo have the angle greater than the angle, the *angle* contained by the equal ſtraight lines. Which was to be demonſtrated.

PROP. XXVI.

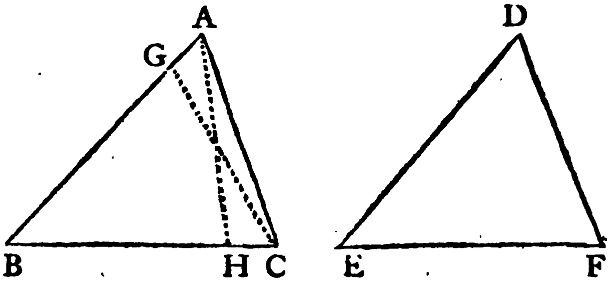
If two triangles have the two angles equal to the two angles, each to each, and one ſide equal to one ſide; either the ſide which is at the equal angles, or that which is extended under one of the equal angles: they will alſo have the *two* remaining ſides equal to the two remaining ſides each to each, and the remaining angle *equal* to the remaining angle.

Let there be two triangles the *triangles* ABC, DEF having the two angles ABC, BCA equal to the two *angles* DEF, EFD each to each; the *angle* ABC equal to the *angle* DEF; and the *angle* BCA *equal* to EFD: and let them have alſo one ſide equal to one ſide:

Firſt, the ſide at the equal angles, the *ſtraight line* BC equal to EF: I ſay alſo that they will have the *two* remaining ſides equal to the *two* remaining ſides, each to each; AB *equal* to DE and AC to DF; and the remaining angle *equal* to the remaining angle, the *angle* BAC *equal* to EDF.

For if AB is unequal to DE, one of them will be greater *than the other*: let AB be the greater and make GB equal (by prop. 3.) to DE, and let CG be joined.

Wherefore becauſe BG is equal to DE (by conſt.) and BC to EF (by ſupp.) certainly the two BG, BC are equal to the two DE, EF each to each; and the angle GBC is equal (by ſupp.) to DEF: therefore (by prop. 4.) the baſe GC is equal to the baſe DF; and the triangle GCB will be equal to the triangle DEF; and the remaining angles will be equal to the remaining angles, each to each, under which the equal ſides are extended: therefore the angle GCB is equal to the *angle* DFE; but the *angle* DFE is ſuppoſed equal to BCA; therefore BCG is equal to BCA the leſs to the greater, which is impoſſible; therefore AB is not unequal to DE; therefore equal: but BC is alſo equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each, and the angle ABC is equal to the angle DEF; wherefore the baſe AC is equal to the baſe DF and the remaining angle BAC is equal to the remaining angle EDF.



But again, let the ſides extended under the equal angles be equal, as AB *equal* to DE: I ſay again that alſo the remaining ſides will be equal to the remaining ſides; AC to DF and BC to EF; and beſides the remaining angle BAC is equal to the remaining angle EDF.

For if BC be unequal to EF, one of them is greater *than the other*; let BC, if it be poſſible, be the greater, and make (by prop. 3.) BH equal to EF, and let AH be joined.

And ſince BH is equal to EF (by conſt.) and AB to DE (by ſupp.); certainly the two AB, BH is equal to the two DE, EF, each to each, and they contain equal angles; therefore (by prop. 4.) the baſe AH is equal to the baſe DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles will be equal to the remaining angles, each to each, under which the equal ſides are extended; wherefore the angle BHA is equal to EFD; but the *angle* EFD is equal (by ſupp.) to the angle BCA; and (by com. not. 1.) BHA is therefore equal to BCA; the outward angle of the triangle AHC viz. BHA equal to the inward and oppoſite

Book I. BCA, which (by prop. 16.) is impoſſible: wherefore BC is not unequal to EF, therefore *it is* equal: but AB is alſo equal to DE; therefore the two AB, BC are equal to the two DE, EF and they contain equal angles; therefore (by prop. 4.) the baſe AC is equal to the baſe DF, and the triangle ABC is equal to the triangle DEF and the remaining angle BAC is equal to the remaining angle EDF.

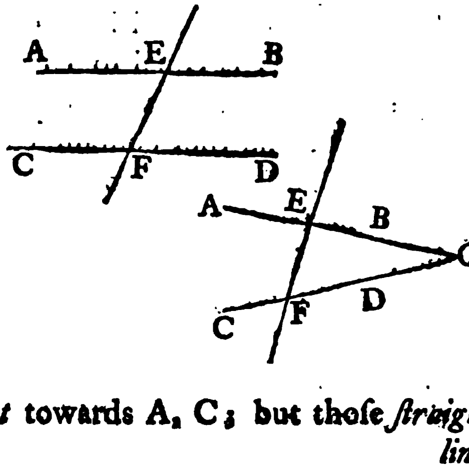
If therefore two triangles have the two angles equal to the two angles, each to each, and have one ſide equal to one ſide; either the ſide which is at the equal angles, or that which is extended under one of the equal angles: they will alſo have the *two* remaining ſides equal to the *two* remaining ſides, each to each, and the remaining angle *equal* to the remaining angle. Which was to be demonſtrated.

PROP. XXVII.

If a ſtraight line falling upon two ſtraight lines makes the alternate angles equal to one another, the ſtraight lines will be parallel to one another.

For let the ſtraight line EF falling upon the two ſtraight lines AB, CD make the alternate angles AEF, EFD equal to one another. I ſay that the *ſtraight line* AB is parallel to the ſtraight line CD.

For if not, AB and CD being produced will meet, either towards the the parts B, D or towards A, C: let them be produced and let them meet towards the parts B, D in the *point* G.



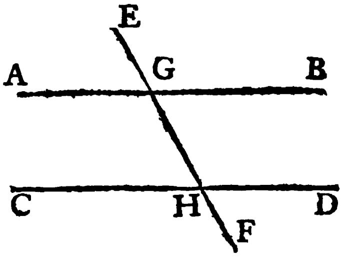
Certainly (by prop. 16.) the outward angle AEF of the triangle EFG is greater than the inward and oppoſite angle EFG; but it is alſo (by ſupp.) equal, which is impoſſible: therefore AB, CD being produced, will not meet towards the parts B, D; certainly in the ſame manner it will be ſhewn, that neither *will they meet* towards A, C; but thoſe *ſtraight* *lines* meeting one another towards neither parts are parallel: therefore AB is parallel to CD.

Wherefore if a ſtraight line falling upon two ſtraight lines make the alternate angles equal to one another, the ſtraight lines will be parallel. Which was to be demonſtrated.

PROP. XXVIII.

If a ſtraight line falling upon two ſtraight lines make the outward angle equal to the inward and oppoſite, and towards the ſame parts; or make the inward angles and towards the ſame parts equal to two right angles: the ſtraight lines will be parallel to one another.

For let the ſtraight line EF falling upon the two ſtraight lines AB, CD make the outward angle EGB equal to the inward and oppoſite and towards the ſame parts the angle GHD; or the inward and towards the ſame parts, the *angles* BGH, GHD equal to two right angles; I ſay that AB is parallel to CD.



For becauſe the *angle* EGB is equal to the *angle* GHD; but EGB is equal (by prop. 15.) to AGH, alſo the *angle* AGH is therefore (by com. not. 1.) equal to GHD: and they are alternate; therefore (by prop. 27.) AB is parallel to CD.

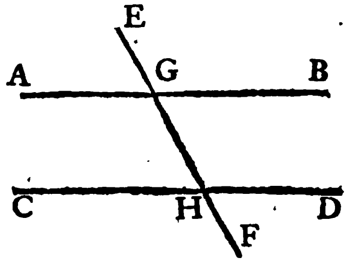
Again becauſe the angles BGH, GHD are (by ſupp.) equal to two right angles; and alſo the *angles* AGH, BGH are equal (by prop. 13.) to two right angles; therefore (by com. not. 1.) the *angles* AGH, BGH are equal to the *angles* BGH, GHD: let the common *angle* BGH be taken away, therefore (by com. not. 3.) the remaining *angle* AGH is equal to the remaining *angle* GHD: and they are alternate; wherefore (by prop. 27.) AB is parallel to CD.

Wherefore if a ſtraight line falling upon two ſtraight lines make the outward angle equal to the inward and oppoſite and towards the ſame parts; or make the inward angles and towards the ſame parts equal to two right angles: the ſtraight lines will be parallel to one another. Which was to be demonſtrated.

PROP. XXIX.

Or a ſtraight line falling upon the two parallel ſtraight lines makes the alternate angles equal to one another; and the outward *angle* equal to the inward and oppoſite and towards the ſame parts; and the inward angles and towards the ſame parts equal to two right angles.

For let the ſtraight line EF fall upon parallel ſtraight lines, the *ſtraight lines* AB, CD: I ſay that it makes the alternate angles AGH, GHD equal; and the outward angle EGB equal to GHD the inward and oppoſite and towards the ſame parts; and BGH, GHD the inward and towards the ſame parts equal to two right angles.



For if AGH be unequal to GHD, one of them is greater: let AGH be the greater: and becauſe AGH is greater than GHD, let BGH *which is* common be added; therefore (by com. not. 4.) the *angles* AGH, BGH are greater than BGH, GHD: But alſo the *angles* AGH, BGH are (by prop. 13.) equal to two right angles; and therefore the angles BGH, GHD are leſs than two right angles; but thoſe *ſtraight lines* which are produced indefinitely from *angles* leſs than two right angles meet *one another* (by com. not. 11.); therefore AB and CD being produced indefinitely will meet *one another*; but they do not meet, becauſe they are ſuppoſed parallel: therefore AGH is not unequal to GHD, therefore *it is* equal.

But the *angle* AGH is equal to the *angle* EGB (by prop. 15.); and therefore (by com. not. 1.) EGB is equal to GHD.

Let the common *angle* BGH be added: wherefore the *angles* EGB, BGH are equal to the *angles* BGH, GHD: but the *angles* EGB, BGH are (by prop. 13.) equal to two right angles; and therefore the angles BGH, GHD are (by com. not. 1.) equal to two right angles.

Or a ſtraight line, therefore, falling upon the two parallel ſtraight lines, makes the alternate angles equal to one another; and the

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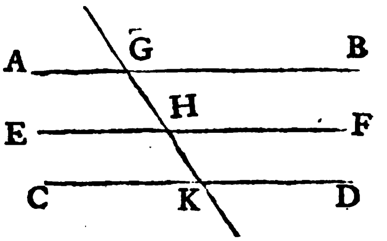
the outward *angle* equal to the inward and oppoſite and towards the ſame parts; and the inward angles and towards the ſame parts equal to two right angles. Which was to be demonſtrated.

PROP. XXX.

Straight lines parallel to the ſame ſtraight line are alſo parallel to one another.

Let each of the *ſtraight lines* AB, CD be parallel to EF: I ſay alſo that AB is parallel to CD.

For let the ſtraight line GK fall upon them.



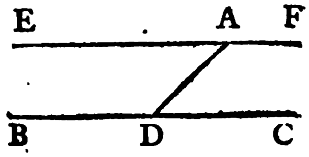
And becauſe the ſtraight line GK hath fallen upon parallel ſtraight lines AB, EF therefore (by prop. 29.) the angle AGH is equal to GHF: again, becauſe the ſtraight line GK hath fallen upon the parallel ſtraight lines EF, CD the angle GHF is equal to GKD: but the angle AGK hath been proved equal to GHF: and therefore (by com. not. 1.) AGK is equal to GKD; and they are alternate *angles*: therefore AB is parallel to CD (by prop. 27.)

Wherefore the *ſtraight lines* which are parallel to the ſame ſtraight line, are parallel to one another. Which was to be demonſtrated.

PROP. XXXI.

Through a given point to draw a ſtraight line parallel to a given ſtraight line

Let the given point be the *point* A; and the given ſtraight line the *ſtraight line* BC; it is required through the point A to draw a ſtraight line parallel to the ſtraight line BC.



Let any point whatever the *point* D be taken in BC; and let AD be joined. And let the *angle* DAE be made, with the ſtraight line AD and at the point A in it, equal to the angle ADC (by prop. 23.): and let the ſtraight line AF be produced in a ſtraight line with AE.

And becauſe the ſtraight line AD falling upon the two ſtraight lines BC, EF hath made the alternate angles, the *angles* EAD, ADC equal to one another, therefore (by prop. 27.) EF is parallel to BC.

Wherefore through a given point the *point* A a ſtraight line EAF hath been drawn parallel to the given ſtraight line BC. Which was to be done.

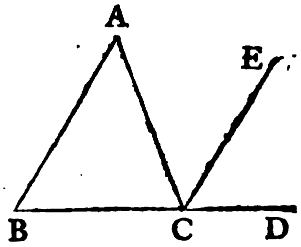
PROP. XXXII.

One of the ſides of any triangle being produced, the outward angle is equal to the two inward and oppoſite *angles:* and the three inward angles of the triangle are equal to two right angles.

Let there be a triangle, the *triangle* ABC, and let one of its ſides BC be produced to the *point* D: I ſay that the outward angle ACD is equal to the two inward and oppoſite *ones* CAB, ABC; and *that* the three inward angles of the triangle viz. ABC, BCA, CAB are equal to two right angles.

For let CE be drawn (by prop. 31.) through the point C parallel to the ſtraight line AB.

And becauſe AB is parallel to CE and AC hath fallen upon them; the alternate angles BAC, ACE are equal (by prop. 27.): Again, becauſe AB is parallel to CE and the ſtraight line BD hath fallen upon them; the outward angle ECD is equal to the inward and oppoſite ABC (by prop. 29.): but ACE has been alſo ſhewn to be equal to BAC; the whole therefore, the outward angle ACD is equal to the two inward and and oppoſite ones BAC, ABC.



Let the common angle ACB be added: therefore the angles ACD, ACB are equal to the three angles ABC, BAC, BCA; but ACD, ACB are (by prop. 13.) equal to two right angles; and therefore (by com. not. 1.) ACB, CBA, CAB are equal to two right angles.

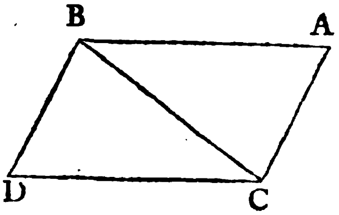
Wherefore one of the ſides of any triangle being produced, the outward angle is equal to the two inward and oppoſite *angles:* and the three inward angles of the triangle are equal to two right angles. Which was to be demonſtrated.

PROP. XXXIII.

The ſtraight lines joining equal and parallel *ſtraight lines* towards the ſame parts, are alſo themſelves equal and parallel.

Let the *ſtraight lines* AB, CD be equal and parallel; and let the ſtraight lines AC, BD join them towards the ſame parts: I ſay that AC, BD are alſo equal and parallel.

For let BC be joined.



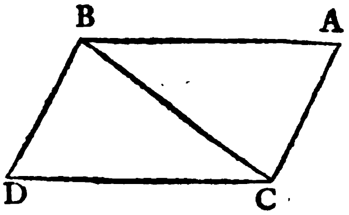
And becauſe AB is parallel to CD, and BC hath fallen upon them; the alternate angles, the *angles* ABC, BCD are (by prop. 29.) equal to one another: and becauſe (by ſupp.) AB is equal to CD, and the *ſtraight line* BC common; certainly the two AB, BC are equal to the two DC, CB; and the angle ABC is equal to BCD; therefore (by prop. 4.) the baſe AC is equal to the baſe BD; and the triangle ABC is equal to the triangle BCD; and the remaining angles will be equal to the remaining angles, under which the equal ſides are extended, each to each; therefore the angle ACB is equal to the *angle* CBD: and becauſe the ſtraight line BC falling upon the two ſtraight lines AC, BD hath made the alternate angles ACB, CBD equal to one another; therefore (by prop. 27.) AC is parallel to BD; and it has been demonſtrated alſo *to be* equal.

Wherefore the ſtraight lines joining equal and parallel *ſtraight lines* towards the ſame parts are alſo themſelves equal and parallel. Which was to be demonſtrated.

PROP. XXXIV.

The oppoſite ſides and alſo the *oppoſite* angles of parallelogram ſpaces are equal to one another, and the diameter cuts the ſpaces in halves.

Let there be a parallelogram, the *parallelogram* ACDB and its diameter the *ſtraight line* BC: I ſay that the oppoſite ſides and alſo the angles of the parallelogram ACDB are equal to one another, and the diameter BC cuts it in halves.



For becauſe AB is parallel to CD and the ſtraight line BC hath fallen upon them; the alternate angles ABC, BCD are equal to one another (by prop. 29.): Again becauſe AC is parallel to BD and BC hath fallen upon them; the alternate angles ACB, CBD are (by prop. 29.) equal to one another: There are two triangles ABC, CBD having the two angles ABC, BCA equal to the two *angles* BCD, CBD, each to each; and one ſide equal to one ſide, the *ſide* at the equal angles, the *ſtraight line* BC common to them both: therefore they will have the remaining ſides equal to the remaining ſides, each to each, (by prop. 26.) and the remaining angle *equal* to the remaining angle; wherefore the ſide AB *is equal* to CD; and AC to BD; and the angle BAC *equal* to the *angle* BDC; and ſince the angle ABC is equal to the *angle* BCD; and the *angle* CBD, to the angle ACB; therefore the whole (by com. not. 2.) angle ACD is equal to the whole ABD; but the angle BAC hath been proved equal to the angle BDC.

Wherefore the oppoſite ſides and alſo the angles of parallelogram ſpaces are equal to one another.

But I ſay alſo that the diameter cuts them in halves: for becauſe AB is equal to CD; and BC common; certainly the two AB, BC are equal to the two DC, CB; each to each; and the angle ABC is equal to the angle BCD; therefore (by prop. 4.) alſo the baſe AC is equal to the baſe BD; and the triangle ABC is equal to the triangle BCD.

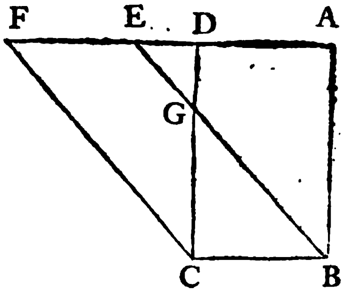
Wherefore the diameter BC cuts in halves the parallelogram ACDB. Which was to be demonſtrated.

PROP. XXXV.

The parallelograms are equal to one another, which are upon the ſame baſe and between the ſame parallel *lines*.

Let there be parallelograms, the *parallelograms* ABCD, EBCF being upon the ſame baſe BC and between the ſame parallel lines AF and BC: I ſay that the *parallelogram* ABCD is equal to the *parallelogram* EBCF.

For



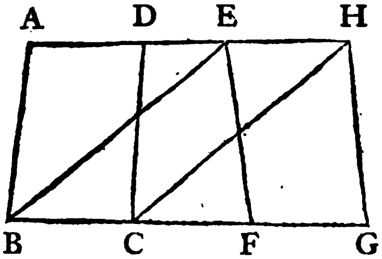
For becauſe ABCD is a parallelogram, AD is (by prop. 34.) equal to BC: and for the ſame reaſon EF is equal to BC; ſo that (by com. not. 1.) AD is equal to EF; and DE is common; therefore (by com. not. 2.) the whole AE is equal to the whole DF; but (by prop. 34.) AB is alſo equal to DC; therefore the two EA, AB are equal to the two FD, DC, each to each; and the angle FDC is equal to the angle EAB (by prop. 29.) the outward to the inward: therefore the baſe EB is equal to the baſe FC; and the triangle EAB is equal to the triangle FCD: let the common *triangle* DEG be taken away; therefore the remainder, the trapezium ABGD is equal to the remainder the trapezium EGCF; let a common part be added, the triangle GBC: therefore (by com. not. 2.) the whole, the parallelogram ABCD is equal to the whole, the parallelogram EBCF.

Therefore the parallelograms, being upon the ſame baſe, and between the ſame parallels are equal to one another. Which was to be demonſtrated.

PROP. XXXVI.

The parallelograms, being upon equal baſes and between the ſame parallels, are equal to one another.

Let there be parallelograms, the *parallelograms* ABCD, EFGH upon equal baſes the *ſtraight lines* BC, FG; and between the ſame parallels AH, BG: I ſay that the parallelogram ABCD is equal to the *parallelogram* EFGH.



For let BE, CH be joined.

And becauſe BC is (by ſupp.) equal to FG, and alſo FG (by prop. 34.) to EH; therefore (by com. not. 1.) BC is equal to EH; but they are likewiſe (by ſup.) parallel; and BE and CH join them; but *ſtraight lines* which join equal and parallel ſtraight lines towards the ſame parts are equal and parallel (by prop. 33.); therefore EB, CH are equal and parallel; wherefore EBCH is a parallelogram, and is equal to

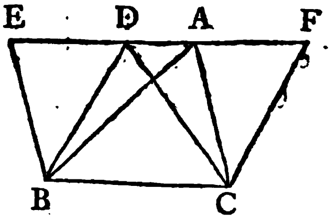
ABCD (by prop. 35.) for it has the ſame baſe with it the *ſtraight line* BC, and it is between the ſame parallels with it the *ſtraight lines* BC, AH. Certainly for the ſame reaſon alſo EFGH is equal to EBCH; therefore (by com. not. 1.) the parallelogram ABCD is equal to EFGH.

Wherefore the parallelograms, being upon equal baſes and between the ſame parallels are equal to one another. Which was to be demonſtrated.

PROP. XXXVII.

The triangles, which are upon equal baſes and between the ſame parallels, are equal to one another.

Let there be triangles, the *triangles* ABC, DBC, being upon the ſame baſe the *ſtraight line* BC; and between the ſame parallels the *ſtraight lines* AD, BC: I ſay that the triangle ABC is equal to the triangle DBC.



Let AD be produced towards both parts to the points E and F; and through the *point* B let BE be drawn parallel to AC; and through the *point* C let CF be drawn parallel to BD.

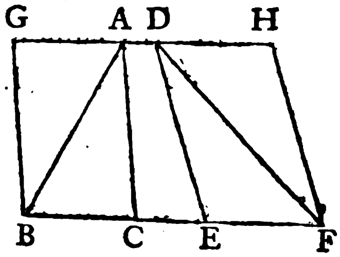
Therefore each of the *figures* EBCA, DBCF is (by conſt. and ſup.) a parallelogram: and EBCA is equal (by prop. 35.) to DBCF for they are upon the ſame baſe BC and between the ſame parallels BC, EF: and the triangle ABC is half of the parallelogram EBCA (by prop. 34.), for the diameter AB cuts it in halves; but the triangle DBC is half of the parallelogram DBCF, for the diameter DC cuts it in halves: but the halves of equal magnitudes are equal; wherefore the triangle ABC is equal to the triangle DBC.

Therefore the triangles, which are upon the ſame baſe, and between the ſame parallels, are equal to one another. Which was to be demonſtrated.

PROP. XXXVIII.

The triangles, which are upon equal baſes and between the ſame parallels, are equal to one another.

Let ABC, DEF be triangles which are upon equal baſes BC, EF; and between the ſame parallels the *ſtraight lines* BF, AD: I ſay that the triangle ABC is equal to the triangle DEF.



For let AD be produced towards both parts to G and H; and through B let BG be drawn parallel to CA; and through the *point* F let FH be drawn parallel to DE.

Therefore (by the conſtruction) each of the *figures* GBCA, DEFH is a parallelogram: and (by prop. 36.) GBCA *is* equal to DEFH, for they are upon equal baſes BC, EF; and between the ſame parallels BF, GH: and (by prop. 34.) the triangle ABC is half of the parallelogram GBCA; for the diameter AB cuts it in halves; alſo the triangle DEF *is* half of the parallelogram DEFH, for the diameter DF cuts it in halves: but (by com. not. 7.) the halves of equal magnitudes are equal: wherefore the triangle ABC is equal to the triangle DEF.

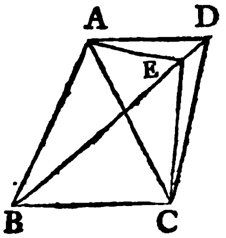
Therefore the triangles, which are upon equal baſes and between the ſame parallels, are equal to one another. Which was to be demonſtrated.

PROP. XXXIX.

The equal triangles, which are upon the ſame baſe, and towards the ſame parts, are between the ſame parallels.

Let ABC, DBC be the equal triangles, which are upon the ſame baſe BC and towards the ſame parts: I ſay that they are between the ſame parallels: for let AD be joined: I ſay that AD is parallel to BC.

For if *it is* not, let AE be drawn (by prop. 31.) through the point A parallel to the ſtraight line BC, and let EC be joined.



Wherefore (by prop. 37.) the triangle ABC is equal to the triangle EBC, for it is upon the ſame baſe with it the *ſtraight line* BC; and between the ſame parallels BC, AE: but the *triangle* ABC is equal (by ſup.) to the *triangle* DBC; therefore (by com. not. 1.) the triangle DBC is equal to the triangle EBC, the greater to the leſs (by com. not. 9.)

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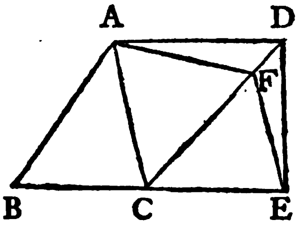
9.) which is impoſſible: wherefore AE is not parallel to BC: certainly we ſhall demonſtrate in like manner, that neither is any other *ſtraight line* except AD; therefore AD is parallel to BC.

Wherefore the equal triangles, which are upon the ſame baſe, and towards the ſame parts, are between the ſame parallels. Which was to be demonſtrated.

PROP. XL.

The equal triangles, which are upon equal baſes, and towards the ſame parts, are between the ſame parallels.

Let ABC, CDE be the equal triangles, which are upon equal baſes the *ſtraight lines* BC, CE; and towards the ſame parts: I ſay that they are between the ſame parallels; for let AD be joined I ſay that AD is parallel to BE.



For if *it is* not; let FA be drawn through the *point* A (by prop. 31.) parallel to BE, and let FE be joined.

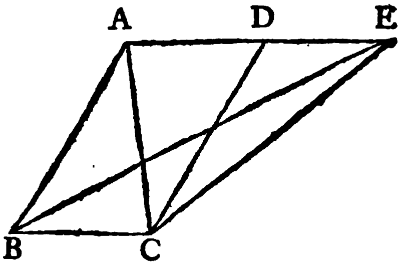
Therefore (by prop. 38.) the triangle ABC is equal to the triangle FCE; for they are upon the equal baſes BC, CE; and between the ſame parallels BE, AF; but (by the ſup.) the triangle ABC is equal to the triangle DCE; and therefore (by com. not. 1.) the triangle DCE is equal to the triangle FCE; the greater to the leſs (by com. not. 9.) which is impoſſible: wherefore AF is not parallel to BE: certainly we ſhall demonſtrate in the ſame manner that neither *is* any other except the *ſtraight line* AD: therefore AD is parallel to BE.

Wherefore the equal triangles, which are upon equal baſes, and towards the ſame parts, are between the ſame parallel lines. Which was to be demonſtrated.

PROP. XLI.

If a parallelogram have the ſame baſe with a triangle; and be between the ſame parallel lines; the parallelogram will be double of the triangle.

For let the parallelogram ABCD have the ſame baſe with the triangle EBC, the *ſtraight line* BC; and let them be between the ſame parallels BC, AE: I ſay that the parallelogram ABCD is double of the triangle BEC.



For let AC be joined.

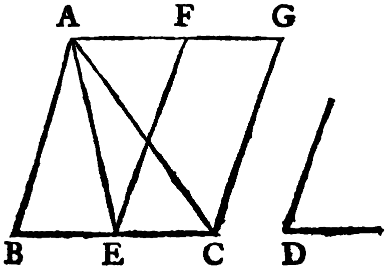
Certainly the triangle ABC is equal to the triangle EBC (by prop. 37.); for it is upon the ſame baſe with it, the *ſtraight line* BC and between the ſame parallels BC, AE; but the parallelogram ABCD (by prop. 34.) is double of the triangle ABC; for the diameter AC cuts it in halves: wherefore the parallelogram ABCD is alſo double of the triangle EBC.

If, therefore, a parallelogram have the ſame baſe with a triangle; and be between the ſame parallels; the parallelogram will be double of the triangle. Which was to be demonſtrated.

PROP. XLII.

To make a parallelogram equal to a given triangle, in a given rectilineal angle.

Let ABC be the given triangle; and D the given rectilineal angle: it is required to make a parallelogram equal to the triangle ABC, in a rectilineal angle equal to D.



Let BC be cut in halves (by prop. 10.) at the *point* E; and let AE be joined; and let the angle CEF be made with the ſtraight line EC and at the point E in it equal to the angle D (by prop. 23.); and through the *point* A let AG be drawn parallel to EC; and through the *point* C (by prop. 31.) let CG be drawn parallel to EF: therefore FECG is a parallelogram.

And becauſe BE is equal to EC, the triangle ABE (by prop. 38.) is alſo equal to the triangle AEC; for they are upon equal baſes

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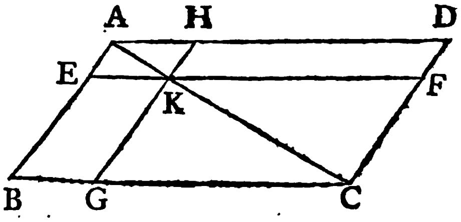
BE, EC and between the ſame parallel *lines* BC, AG; wherefore the *triangle* ABC is double of the triangle AEC: but the parallelogram FECG is alſo (by prop 41.) double of the triangle AEC: for they have the ſame baſe, and are between the ſame parallels; therefore (by com. not. 6.) the parallelogram FECG is equal to the triangle ABC; and has the angle CEF equal to the given *angle* D.

Therefore a parallelogram the *parallelogram* FECG has been made equal to the given triangle ABC, in an angle the *angle* CEF, which is equal to the *angle* D. Which was to be done.

PROP. XLIII.

The complements of the parallelograms about the diameter of every parallelogram are equal to one another.

Let ABCD be a parallelogram, and AC the diameter of it; and let EH, FG be the parallelograms about AC; and the *parallelograms* BK, KD thoſe which are called complements: I ſay that the complement BK is equal to the complement KD.



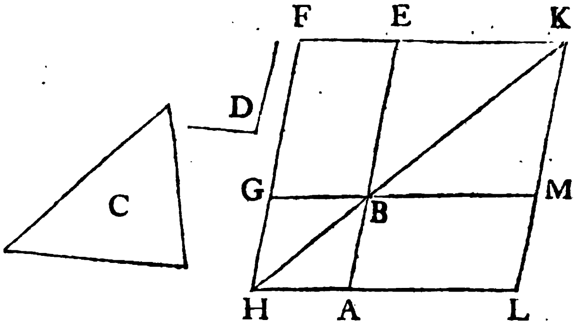
For ſince ABCD is a parallelogram, and AC the diameter of it; the triangle ABC is (by prop. 34.) equal to the triangle ADC: again becauſe EKHA is a parallelogram, and AK the diameter of it; the triangle AEK is (by prop. 34.) equal to the triangle AHK: certainly for the ſame reaſon alſo the triangle KFC is equal to the *triangle* KGC: becauſe therefore the triangle AEK is equal to the triangle AHK; and KFC to KGC; the triangle AEK together with the triangle KGC is equal (by com. not. 2.) to the triangle AHK together with the triangle KFC; but the whole triangle ABC is equal to the whole triangle ADC; therefore (by com. not. 3.) the *remainder*, the complement BK is equal to the remainder the complement KD.

Wherefore the complements of the parallelograms about the diameter of every parallelogram are equal to one another. Which was to be demonſtrated.

PROP. XLIV.

To apply a parallelogram, to a given ſtraight line, equal to a given triangle, in a given rectilineal angle.

Let the given ſtraight line be AB; but the given triangle the *triangle* C; and the given rectilineal angle, the *angle* D; it is required to apply a parallelogram to the given ſtraight line AB, equal to the given triangle C; in an angle equal to D.



Let the parallelogram BEFG be made (by prop. 42.) equal to the triangle C, in an angle EBG which is equal to D: and let it be placed in ſuch a manner that BE may be in a ſtraight line with AB; and let FG be produced to H; and through the *point* A, let AH be drawn (by prop. 31.) parallel to either of the *ſtraight lines* BG or EF; and let HB be joined.

And becauſe the ſtraight line HF hath fallen upon the parallels AH, EF; therefore the angles (by prop. 29.) AHF, HFE are equal to two right angles: wherefore the angles BHG, GFE are leſs than two right angles: but (by com. not. 11.) thoſe *ſtraight lines*, which are produced indefinitely from angles leſs than two right angles, do meet *each other*; therefore HB, FE being produced will meet: let them be produced, and let them meet in the *point* K; and through the point K, let KL be drawn parallel to either of the *lines* EA, FH; and let GB, HA be produced to the points M, L.

Therefore HLKF is a parallelogram, and HK its diameter; and AG, ME *are* the parallelograms about HK; and LB, BF the *parallelograms* called complements: wherefore (by prop. 43.) LB is equal to BF: but BF is alſo equal to the triangle C; and therefore (by com. not. 1.) LB is equal to C: and ſince the angle GBE is equal (by prop. 15.) to ABM, but GBE is equal (by conſt.) to D; alſo (by com. not. 1.) ABM is equal to the angle D.

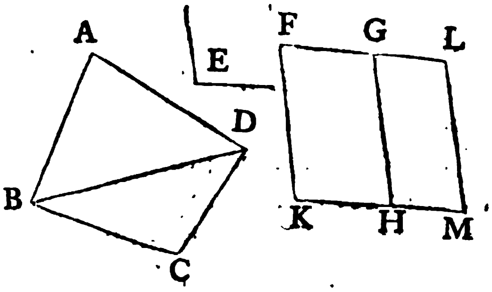
Wherefore a parallelogram LB hath been applied to the given ſtraight line AB, equal to the given triangle C; in an angle ABM, which is equal to the *angle* D. Which was to be done.

PROP. XLV.

To make a parallelogram equal to a given rectilineal *figure* in a given rectilineal angle.

Let the given rectilineal *figure* be ABCD; and E the given rectilineal angle: it is required to make a parallelogram equal to the rectilineal *figure* ABCD in an angle equal to E.

For let DB be joined; and (by prop. 42.) let the parallelogram FH be made equal to the triangle ABD, in the angle HKF, which is equal to E; and (by prop. 44.) let the parallelogram GM, equal to the triangle DBC, be applied to the ſtraight line GH, in the angle GHM, which is equal to E.



And becauſe the angle E is equal to either of the angles HKF, GHM; therefore alſo (by com. not. 1.) HKF is equal to GHM: let the common angle KHG be added; therefore (by com. not. 2.) FKH, KHG are equal to KHG, GHM: but (by prop. 29.) FKH, KHG are equal to two right angles; wherefore alſo KHG, GHM are equal to two right angles: to a certain ſtraight line GH, and to a point in it H, the two ſtraight lines KH, HM not lying towards the ſame parts, make the adjacent angles equal to two right angles; therefore (by prop. 14.) KH is in a ſtraight line with HM: and ſince the ſtraight line GH hath fallen upon the parallels KM, FG; the alternate angles MHG, HGF are equal (by prop. 29.): let the common angle HGL be added: therefore the angles MHG, HGL are equal to HGF, HGL; but (by prop. 29.) MHG, HGL are equal to two right angles; therefore alſo HGF, HGL are equal to two right angles; wherefore (by prop. 14.) FG is in a ſtraight line with GL. And becauſe KF is equal and alſo parallel to GH; but GH *is equal and alſo parallel* to ML; therefore (by com. not. 1. and prop. 30.) KF is equal and parallel to ML; and the ſtraight lines KM, FL join them: and (by prop. 33.) KM, FL are equal and parallel. Wherefore KFLM is a parallelogram; and becauſe the triangle ABD is equal to the parallelogram HF, and the *triangle* DBC to the *parallelogram* GM; therefore the whole rectilineal *figure* ABCD is equal to the whole parallelogram KFLM.

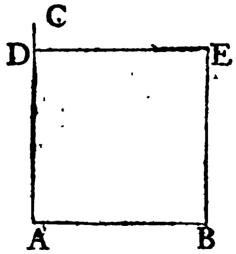
Wherefore a parallelogram KFLM is made equal to a given rectilineal *figure* ABCD in an angle, the *angle* FKM, which is equal to the given *angle* E. Which was to be done.

PROP. XLVI.

To deſcribe a ſquare upon a given ſtraight line.

Let the given ſtraight line be AB: it is required to deſcribe a ſquare upon the ſtraight line AB.

Let AC be drawn (by prop. 11,) at right angles to the ſtraight line AB from the point A in it; and (by prop. 3.) make AD equal to AB: and let DE be drawn, (by prop. 31.) through the point D, parallel to AB; and let BE be drawn through the point B parallel to AD.



Therefore the *figure* ADEB is a parallelogram; wherefore (by prop. 34.) AB is equal to DE; and AD to BE; but alſo AB is equal to AD (by conſt.) therefore the four *ſtraight lines* BA, AD, DE, EB are equal to one another: wherefore the parallelogram ADEB is equilateral; I ſay that it is alſo rectangular: For becauſe the ſtraight line AD has fallen upon the two parallels AB, DE; therefore (by prop. 29.) the angles BAD, ADE are equal to two right angles; but BAD is a right angle (by conſt.); therefore ADE is a right angle; but the oppoſite ſides and alſo the angles of parallelogram ſpaces are equal to one another (by prop. 34.); therefore each of the oppoſite angles is a right angle, the *angles* ABE, BED; wherefore the *figure* ADEB is rectangular; but it has been demonſtrated to be equilateral alſo.

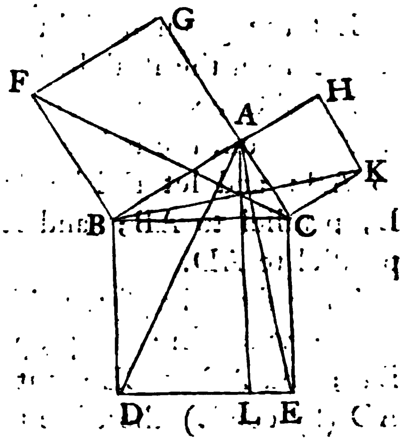
Therefore (by def. 30.) it is a ſquare; and it hath been deſcribed upon the ſtraight line AB. Which was to be done.

PROP. XLVII.

In right angled triangles, the ſquare of the ſide ſubtending the right angle, is equal to the ſquares of the ſides containing the right angle.

Let ABC be a right angled triangle, having the angle BAC a right *angle:* I ſay that the ſquare of *BC* is equal to the ſquares of BA, AC.

For (by prop. 46.) let the ſquare BDEC be deſcribed upon BC; and the *ſquares* GB, HC upon BA, AC, and through the *point* A let AL be drawn parallel either to BD or CE; and let AD, FC be joined.



And becauſe each of the angles BAC, BAG is (by conſt. and ſup.) a right *angle*; to a certain *line* BA, and to the point A in it, two ſtraight lines AC, AG, not lying towards the ſame parts, make the adjacent angles equal to two right angles; therefore (by prop. 14.) AC is in a ſtraight line with AG: certainly for the ſame reaſon alſo AB is in a ſtraight line with AH; and ſince the angle DBC is equal to the *angle* FBA (by com. not. 10.), for each *of them* is a right angle, let the common *angle* ABC be added; therefore (by com. not. 2.) the whole *angle* DBA is equal to the whole *angle* FBC; and becauſe the two *ſtraight lines* DB, BA are equal to the two CB, BF, each to each *(being ſides of the ſame ſquare)* and the angle DBA is equal to the angle FBC; therefore (by prop. 4.) the baſe AD is equal to the baſe FC, and the triangle ABD is equal to the triangle FBC; and the parallelogram BL is double of the triangle ABD (by prop. 41.); for they have the ſame baſe BD and are between the ſame parallels BD, AL: but the ſquare GB is double of the triangle FBC; for again, they have the ſame baſe FB, and are between the ſame parallels FB, GC; but the doubles of equal magnitudes are equal to one another (by com. not 6.) wherefore alſo the parallelogram BL is equal to the ſquare GB: Certainly in the ſame manner it will be demonſtrated, having joined AE, BK, that the parallelogram CL is equal to the ſquare HC; therefore the whole ſquare DBCE is equal to the two ſquares GB, HC; and BDEC is the ſquare deſcribed upon BC; and GB, HC the *ſquares deſcribed* upon BA, AC: wherefore the ſquare BE *deſcribed* upon the ſide BC is equal to the ſquares upon the ſides BA, AC.

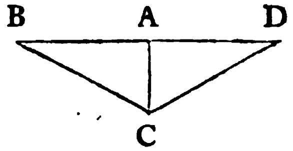
Wherefore in right angled triangles, the ſquare of the ſide ſubtending the right angle is equal to the ſquares of the ſides, containing the right angle. Which was to be demonſtrated.

PROP. XLVIII.

If the ſquare of one of the ſides of a triangle be equal to the ſquares of the two remaining ſides of the triangle; the angle contained by the two remaining ſides of the triangle is a right angle.

For let the ſquare of BC one ſide of the triangle ABC be equal to the ſquares of the ſides BA, AC: I ſay that the angle BAC is a right angle.

For from the point A let the ſtraight line AD be drawn at right angles to AC; and make AD (by prop. 3.) equal to BA; and join DC.



And becauſe DA is equal to AB, the ſquare of AD is equal to the ſquare of AB; let the common ſquare of AC be added; therefore (by com. not. 2.) the ſquares of DA and AC are equal to the ſquares of BA and AC: but the ſquares of DA and AC are (by prop. 47.) equal to the ſquare of DC; for DAC is (by conſt.) a right angle; and the ſquare of BC is equal to the ſquares of BA and AC, for this is ſuppoſed: wherefore (by com. not. 1.) the ſquare of DC is equal to the ſquare of BC; ſo that alſo the ſide DC is equal to BC; and ſince AD is equal to AB and AC common; certainly the two AD, AC are equal to

Book I. to the two AB, AC, and the baſe DC is equal to the baſe BC; therefore the angle DAC is equal to the angle BAC; but DAC is a right angle; therefore the *angle* BAC is a right angle.

Wherefore if the ſquare of one of the ſides of a triangle be equal to the ſquares of the two remaining ſides of the triangle; the angle contained by the two remaining ſides of the triangle is a right angle. Which was to be demonſtrated.

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THE

ELEMENTS

OF

EUCLID.

BOOK II.

DEFINITIONS.

Book II.

1. EVERY right angled parallelogram is ſaid to be contained by the two ſtraight lines containing the right angle.

2. Of every parallelogram ſpace, let any one of the parallelograms about the diameter of it, together with the two complements, be called a GNOMON.

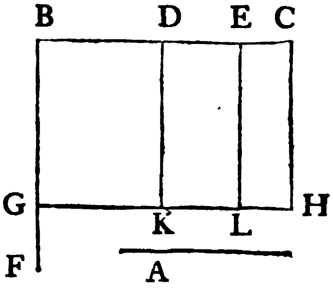
PROP. I.

If there be two ſtraight lines, and one of them be cut into any number of ſegments; the rectangle contained by the two ſtraight lines is equal to the rectangles contained by the undivided line and each of the ſegments *of the divided line.*

Let there be two ſtraight lines A, BC, and let BC be cut as it may happen in the points D and E: I ſay that the rectangle contained

tained by A and BC is equal to the rectangle contained by A and BD, and to that *contained* by A and DE, and beſides to that *contained* by A and EC.

For let BF be drawn from the *point* B at right angles to BC; and let BG be made (by prop. 3. B. 1.) equal to A; and let GH be drawn, through the point G parallel to BC (by prop. 31. B. 1.); and let DK, EL, CH be drawn through the points D, E, C parallel to BG.



Certainly the *rectangle* BH is equal to the rectangles BK, DL, EH; and BH is the rectangle *contained* by A and BC; for (by def. 1. B. 2.) it is contained by GB, BC; but BG is (by conſt.) equal to A: and BK is *a rectangle contained* by A and BD; for (by def. 1.) it is contained by GB, BD; and GB is equal to A: but the rectangle DL is *contained* by A and DE; for DK, that is BG (by prop. 34. B. 1.) is equal to A: and alſo in like manner EH is *a rectangle* contained by A and EC, therefore the rectangle contained by A and BC is equal to the *rectangles contained* by A and BD; and A and DE and alſo A and EC.

Wherefore if there be two ſtraight lines, and one of them be cut into any number of ſegments; the rectangle contained by the two ſtraight lines is equal to the rectangles contained by the undivided line and each of the ſegments *of the divided line.* Which was to be demonſtrated.

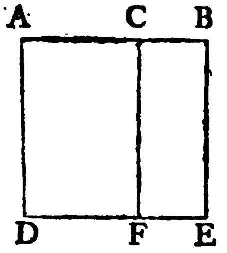
PROP. II.

If a ſtraight line be cut as it may happen, the rectangles contained by the whole and each of the ſegments are equal to the ſquare of the whole *line.*

For let the ſtraight line AB be cut as it may happen in the point C: I ſay that the rectangle *contained* by AB and BC; together with the rectangle contained by BA and AC is equal to the ſquare of AB.

For let the ſquare ADEB be deſcribed upon AB (by 46. 1.); and let CF be drawn (by 31. 1.) parallel either to AD or BE.

Certainly



Certainly the *rectangle* AE is equal to the *rectangles* AF and CE: and the *rectangle* AE is the ſquare of AB; and the *rectangle* AF is a rectangle contained by BA, AC; for (by def. 1. 2.) it is contained by DA, AC; and AD is equal to AB (being ſides of a ſquare): and the rectangle CE is contained by AB, BC; for EB is equal to AB: wherefore the rectangle contained by BA, AC together with that contained by AB, BC is equal to the ſquare of AB.

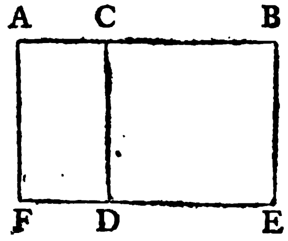
If therefore a ſtraight line be cut as it may happen, the rectangles contained by the whole and each of the ſegments are equal to the ſquare of the whole *line.*

PROP. III.

If a ſtraight line be cut as it may happen, the rectangle contained by the whole and one of the ſegments is equal to the rectangle contained by the *two* ſegments; and to the ſquare of the forementioned ſegment.

For let the ſtraight line AB be cut as it may happen in the point C: I ſay that the rectangle contained by AB, BC is equal to the rectangle contained by AC, CB; together with the ſquare of BC.

For let CDEB the ſquare of BC be deſcribed (by 46. 1.); and let ED be produced to F: and through the *point* A let AF be drawn (by 31. 1.) parallel to either of the *ſtraight lines* CD, BE.



Certainly the rectangle AE is equal to the *rectangles* AD, CE; and the *rectangle* AE is a rectangle contained by AB, BC; for (by def. 1. B. 2.) it is contained by AB, BE; but BE is equal to BC (being ſides of a ſquare): And the rectangle AD is contained by AC, CB; for DC is equal to CB; but the *rectangle* DB is the ſquare of CB: therefore the rectangle contained by AB, CB is equal to the rectangle contained by AC, CB; together with the ſquare of BC.

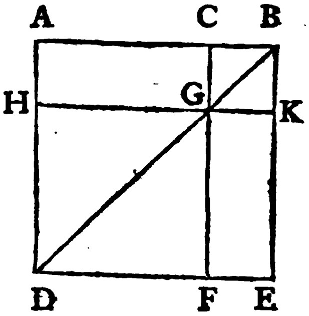
If therefore a ſtraight line be cut as it may happen, the rectangle, contained by the whole and one of the ſegments, is equal to the rectangle contained by the *two* ſegments; and to the ſquare of the forementioned ſegment. Which was to be demonſtrated.

PROP. IV.

If a ſtraight line be cut as it may happen, the ſquare of the whole *line* is equal to the ſquares of the *two* ſegments, and to the rectangle, contained by the *two* ſegments, *taken* twice.

For let the ſtraight line AB be cut as it may happen in the *point* C: I ſay that the ſquare of AB is equal to the ſquares of AC, CB and the rectangle contained by AC, CB *taken* twice.

For let the ſquare ADEB be deſcribed upon AB (by 46. 1.); and let BD be joined, and through the *point* C let CGF be drawn parallel to either of the *lines* AD, BE; and through the point G let HK be drawn parallel to either of the *lines* AB, DE.



And becauſe CF is parallel to AD and BD hath fallen upon them; the outward angle BGC is equal to the inward and oppoſite ADB: but the angle ADB (by 5. 1.) is equal to ABD, becauſe the ſide BA is equal to AD; therefore (by com. not. 1.) the angle CGB is equal to GBC; ſo that alſo (by 6. 1.) the ſide BC is equal to the ſide CG: but (by 34. 1.) CG is equal to BK and CB to GK; wherefore alſo GK is equal to KB: wherefore CGKB is equilateral; I ſay alſo that it is rectangular: For becauſe CG is parallel to BK and CB hath fallen upon them; therefore (by 29. 1.) the angles KBC, BCG are equal to two right angles; but KBC is a right angle; wherefore alſo GCB *is* a right angle; ſo that alſo (by 34. 1.) the oppoſite *angles* CGK, GKB are right angles: wherefore the *figure* CGKB is rectangular; and it has been demonſtrated to be equilateral; therefore (by def. 30. 1.) it is a ſquare and it *is deſcribed* upon BC: Certainly for the ſame reaſon alſo *the figure* HF is a ſquare; and it is *deſcribed* upon HG that is upon AC: therefore the *figures* HF, CK are the ſquares of AC, CB. And becauſe AG is equal to GE (by 43. 1.); and AG is the *rectangle contained* by AC, CB; for GC is

GC is equal to CB; and GE is therefore equal to the *rectangle contained* by AC, CB: wherefore AG, GE are equal to the *rectangle contained* by AC, CB *taken* twice; but alſo HF, CK are the ſquares of AC, CB; therefore the four *figures* HF, CK, AG, GE are equal to the ſquares of AC, CB and the *rectangle contained* by AC, CB *taken* twice: but HF, CK, AG, GE are the whole *figure* ADEB, which is the ſquare of AB; wherefore the ſquare of AB is equal to the ſquares of AC, CB and the *rectangle contained* by AC, CB *taken* twice.

Wherefore if a ſtraight line be cut as it may happen, the ſquare of the whole *line* is equal to the ſquares of the *two* ſegments, and to the *rectangle* contained by the *two* ſegments *taken* twice.

Another Demonſtration.

I ſay that the ſquare of AB is equal to the ſquares of AC, CB and to the rectangle contained by AC, CB *taken* twice.

For, making uſe of the ſame figure, becauſe BA is equal to AD, (by 5. 1.) the angle ABD is equal to the *angle* ADB; and ſince (by 32. 1.) the three angles of every triangle are equal to two right angles; therefore the three angles of the triangle ABD viz. ABD, ADB, BAD are equal to two right angles; but BAD is a right angle (being an angle of a ſquare); therefore the remaining angles ABD, ADB are equal to one right angle; and they are equal: each, therefore, of the angles ABD, ADB is the half of a right angle: but BCG is a right angle (by prop. 29. 1.) for it is equal to the inward and oppoſite *angle* at A; therefore the remainder CGB is the half of a right angle: wherefore the angle CGB is equal to CBG; ſo that alſo (by 6. 1.) the ſide BC is equal to the *ſide* CG; but (by 34. 1.) CB is equal to KG, and CG to BK: wherefore the *figure* CK is equilateral; but it has a right angle, the *angle* CBK: wherefore CK is a ſquare; and it is *deſcribed* upon CB. Certainly for the ſame reaſon alſo HF is a ſquare; and is equal to the ſquare of AC: wherefore CK, HF are ſquares, and are equal to the ſquares of AC, CB: and becauſe AG is equal to GE (by 43. 1.) and AG is the *rectangle contained* by AC, CB; for CG is equal to CB; and EG is equal to the *rectangle contained* by AC, CB; therefore the *figures* AG, GE are equal to the *rectangle con-*

tained by AC, CB; *taken* twice: but CK, HF are equal to the *ſquares* of AC, CB; therefore the *figures* CK, HF, AG, GE are equal to the *ſquares* of AC, CB and the *rectangle contained* by AC, CB *taken* twice: But the *ſquares* CK, HF and the rectangles AG, GE are the whole *ſquare* AE, which is the ſquare of AB.

Wherefore the ſquare of AB is equal to the ſquares of AC, CB; and to the rectangle contained by AC, CB *taken* twice. Which was to be demonſtrated.

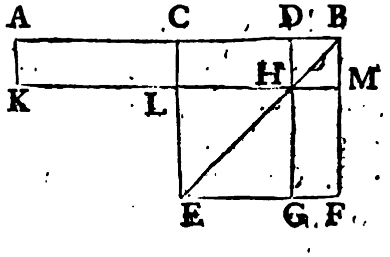
Cor. Certainly from theſe *demonſtrations* it is manifeſt that in ſquare ſpaces the parallelograms about the diameter are ſquares.

PROP. V.

If a ſtraight line be cut into equal and unequal *ſegments*; the rectangle contained by the unequal ſegments of the whole *line*, together with the ſquare of the *line* between the *points* of ſection is equal to the ſquare of half *the line*.

For let any ſtraight line AB be cut into equal *ſegments* at the *point* C; and into unequal *ſegments* at the *point* D; I ſay that the rectangle contained by AD, DB together with the ſquare of CD is equal to the ſquare of CB.

For let the ſquare CEFB be deſcribed upon BC; and let BE be joined; and let DHG be drawn, through the *point* D, parallel to either of the *lines* CE, BF; and again through the point H let KLM be drawn parallel to either of the *lines* CB, EF; and again through the *point* A let AK be drawn parallel to either of the *lines* CL, BM.



And ſince the complement CH is equal to the complement HF; let DM *which is* common be added; therefore the whole CM is equal to the whole DF: but CM is equal to AL (by prop. 36. 1.) becauſe AC is equal to CB: therefore alſo (by com. not. 1.) AL is equal to DF; let CH *which is* common be added; therefore the whole AH is equal to DF and DL: but AH is the *rectangle contained* by AD, DB for DH is equal to DB (by cor. to 4. 2.): but

FD, DL or FDL is called a Gnomon (by def. 2. 2.); therefore the Gnomon FDL is equal to the *rectangle contained* by AD, DB: Let LG which is common be added; which (by Cor. to 4. 2.) is equal to the ſquare of CD: wherefore the Gnomon FDL and the *ſquare* LG is equal to the rectangle contained by AD, DB and the ſquare of CD: but the Gnomon FDL and the ſquare LG is the whole ſquare CEFB, which is the *ſquare* of CB: Wherefore the rectangle contained by AD, DB together with the ſquare of CD is equal to the ſquare of CB.

Wherefore if a ſtraight line be cut into equal and unequal *ſegments*; the rectangle contained by the unequal ſegments of the whole *line*, together with the ſquare of the *line* between the *points* of ſection, is equal to the ſquare of half *the line*. Which was to be demonſtrated.

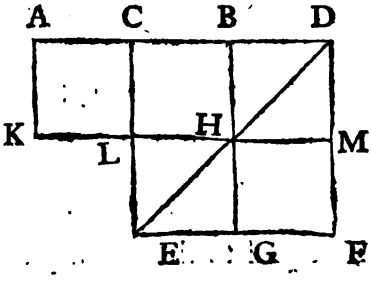
PROP. VI.

If a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the rectangle contained by the whole with the part produced, and the part produced, together with the ſquare of half the *line*; is equal to the ſquare deſcribed upon the compounded *line*, that is of half the line and the part produced, as upon one *line*.

For let any ſtraight line AB be cut in halves at the point C, and let any ſtraight line BD be added to it in a ſtraight line: I ſay that the rectangle contained by AD, DB together with the ſquare of BC is equal to the ſquare of CD.

For let the ſquare CEFD be deſcribed (by 46. 1.) upon CD; and let DE be joined; and through the point B let (by 31. 1.) BHG be drawn parallel to either of the *lines* CE, DF; and through the point H, let KLM be drawn parallel to either of the *lines* AD, EF; and farther through the *point* A let AK be drawn parallel to either of the *lines* CL, DM.

Therefore becauſe AC is equal to CB, the *rectangle* AL (by prop. 36. 1.) is equal to the *rectangle* CH; but CH is alſo equal (by 43. 1.) to HF; wherefore AL is equal to HF; let CM *which is* common be added, therefore the whole AM is equal to CM, MG that is (by def. 2. 2.)



to the Gnomon CMG: but AM is the *rectangle contained* by AD, DB (by def. 1.2.) for DM is equal to DB (by cor. to 4.2.); therefore the Gnomon CMG is equal to the rectangle contained by AD, DB: let GL *which is* common be added; which is equal (by cor. to 4. 2.) to the ſquare of CB: therefore the rectangle contained by AD, DB together with the ſquare of CB is equal to the Gnomon CMG and the *ſquare* LG: but the Gnomon CMG and the *ſquare* LG is the whole ſquare CEFD, which is the ſquare of CD: wherefore the rectangle contained by AD, DB together with the ſquare of CB is equal to the ſquare of CD.

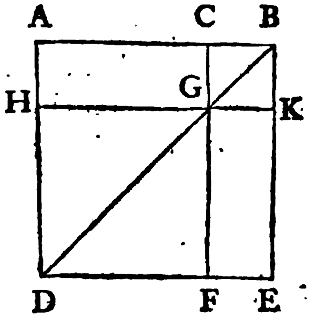
Wherefore if a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the rectangle contained by the whole with the part produced and the part produced; together with the ſquare of half of the *line*; is equal to the ſquare deſcribed upon the compounded *line*; that is of half *the line* and the part produced, as upon one *line*. Which was to be demonſtrated.

PROP. VII.

If a ſtraight line be cut as it may happen; the *ſquare* of the whole *line* and the *ſquare* of one of the ſegments, theſe ſquares, taken together, are equal to the rectangle contained by the whole, and the ſaid ſegments, *taken* twice and the ſquare of the remaining ſegment.

For let any ſtraight line AB be cut as it may happen in the point C; I ſay that the ſquares of AB, BC are equal to the rectangle contained by AB, BC taken twice and the ſquare of AC.

For let a ſquare ADEB be deſcribed upon AB: and let the figure be conſtructed.



And becauſe AG is equal to GE (by 43. 1.); let CK which is common be added: therefore the whole AK is equal to the whole CE; therefore AK, CE are the double of AK: but AK, CE is the Gnomon AKF: and the ſquare CK: therefore the Gnomon AKF and the ſquare CK are double of the rectangle AK: but the double of AK is alſo equal to the rectangle contained by AB, BC *taken* twice; for BK is (by cor. to 4. 2.) equal to BC: Wherefore the Gnomon AKF and the ſquare CK are equal to the rectangle contained by AB, BC *taken* twice: Let HF which is common be added, which is the ſquare of AC: therefore the Gnomon AKF and the ſquares CK and HF are the whole *ſquare* ADEB and the *ſquare* CK, which are the ſquares of AB, BC: wherefore the ſquares of AB, BC are equal to the rectangle contained by AB, BC *taken* twice together with the ſquare of AC.

Wherefore if a ſtraight line be cut as it may happen; the *ſquare* of the whole *line* and the ſquare of one of the ſegments, theſe ſquares, taken together, are equal to the rectangle contained by the whole and the ſaid ſegment, taken twice; and the ſquare of the remaining ſegment. Which was to be demonſtrated.

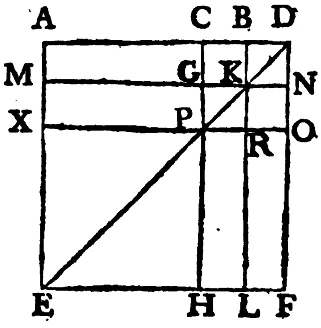
PROP. VIII.

If a ſtraight line be cut as it may happen; the rectangle contained by the whole *line* and one of the ſegments, *taken* four times, together with the ſquare of the remaining ſegment, is equal to the ſquare deſcribed upon the whole and the ſaid ſegment, as upon one *line*.

For let any ſtraight line AB be cut as it may happen in the point C: I ſay that the rectangle contained by AB, BC; *taken* four times, together with the ſquare of AC is equal to the ſquare deſcribed upon AB, BC; as upon one line.

For let BD be produced in a ſtraight line with AB; and make BD equal to CB; and let the ſquare AEFD be deſcribed upon AD: and let the double figure be conſtructed.

Therefore becauſe CB is equal to BD; but (by 34. 1.) CB is equal to GK, and BD to KN; alſo (by com. not. 1.) GK is equal to KN. Certainly for the ſame reaſon PR is equal to RO: and becauſe CB is equal to BD, and GK to KN; therefore alſo the *figure* CK is equal to BN (by 36. 1.), and GR to RN; but the *figure* CK is equal to RN (by 43. 1.); for they



are the complements of the parallelogram CO: wherefore BN is alſo equal to GR (by con. not. 1.): therefore the four *figures* the *rectangles* BN, KC, GR, RN are equal to one another; the four therefore are quadruple of CK: Again becauſe CB is equal to BD; but BD (by cor. to 4. 2.) is equal to BK, that is to CG (by 34. 1.); and CB to GK, that is, is equal to GP; therefore CG is equal to GP: And becauſe CG is equal GP; and PR to RO; alſo the *rectangle* AG is equal (by 36. 1.) to MP: and PL to RF; but MP and PL are equal (by 43. 1.), for they *are* the complements of the parallelogram ML: therefore alſo AG is equal to RF: wherefore the four *figures*, the rectangles AG, MP, PL, RF are equal to one another: therefore the four *figures* are quadruple of AG: but it has been demonſtrated that the four CK, BN, GR, RN are quadruple of CK; therefore the eight which contain the Gnomon AOH are quadruple of AK; and becauſe AK is the *rectangle contained* by AB, BD; for BK is equal to BD; therefore the rectangle contained by AB, BD *taken* four times is equal to the Gnomon AOH: let XH which is common be added, which is equal to the ſquare of AC: wherefore the rectangle contained by AB, BD *taken* four times, together with the ſquare of AC is equal to the Gnomon AOH and the *ſquare* XH: but the Gnomon AOH and the *ſquare* XH is the whole ſquare AEFD, which is the *ſquare* of AD: therefore the rectangle contained by AB, BC *taken* four times together with the ſquare of AC is equal to the ſquare of AD, that is to the ſquare deſcribed upon AB and BC as upon one *line*.

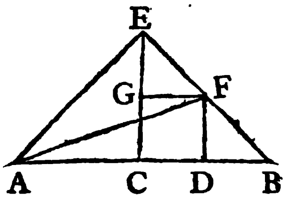
Wherefore if a ſtraight line be cut as it may happen; the rectangle contained by the whole *line*, and one of the ſegments; taken four times; together with the ſquare of the remaining ſegment, is equal to the ſquare deſcribed upon the whole and the ſaid ſegment, as upon one *line*. Which was to be demonſtrated.

PROP. IX.

If a ſtraight line be cut into equal and unequal *ſegments*; the ſquares, of the unequal ſegments of the whole *line*, is double both of the ſquare of half *the line* and of the *line* between the *points* of ſection.

For let the ſtraight line AB be cut into equal ſegments at the *point* C, and into unequal ones at the point D: I ſay that the ſquares of AD, DB are double of the ſquares of AC, CD.

For from the *point* C, let CE be drawn at right angles to AB; and let it be made equal to either of the lines AC, CB; and let EA, EB be joined; and through the point D let DF be drawn parallel to EC; and through the point F, let FG be drawn parallel to AB, and let AF be joined.



And becauſe AC is equal to CE, the angle EAC (by 5. 1.) is equal to AEC: and ſince the angle at C is a right angle, therefore the remaining angles AEC, EAC are equal to one right angle (by 32. 1.): therefore each of the angles AEC, EAC is the half of a right angle. Certainly for the ſame reaſon alſo each of the *angles* CEB, EBC is the half of a right angle; wherefore the whole angle AEB is a right angle: and becauſe GEF is half a right angle, and EGF a right angle (by. 29. 1.) for it is equal to the inward and oppoſite ECB; therefore the remainder EFG is half a right angle (by 32. 1.); therefore the angle GEF is equal to the *angle* EFG; ſo that alſo (by 6. 1.) the ſide EG is equal to the ſide GF: Again becauſe the angle at B is half a right angle, and FDB a right angle (by 29. 1.) for again it is equal to the inward and oppoſite ECB; therefore the remaining *angle* BFD is half a right angle: therefore the angle at B is equal to the angle DFB; ſo that alſo the ſide DF is equal (by 6. 1.) to the ſide DB: And becauſe AC is equal to CE: the *ſquare* of AC is alſo equal to the *ſquare* of CE; therefore the ſquares of AC, CE; are double of the ſquare of AC: but the ſquare of EA is equal to the *ſquares* of AC, CE (by 47. 1.) for the angle ACE is a right angle: wherefore the ſquare of EA is double of the ſquare of AC: Again becauſe EG is equal to GF, the *ſquare* of EG is equal to the *ſquare* of GF; therefore the ſquares of EG, GF are double of the ſquare of GF; but the ſquare of EF is equal to the ſquares of EG, GF (by 47. 1.); therefore the ſquare of EF is double of the ſquare of GF; but (by 34. 1.) GF is equal to CD; therefore the *ſquare* of EF is double the *ſquare* of CD: but the *ſquare* of AE is alſo the double of the *ſquare* of AC; wherefore the ſquares of AE, EF are double of the ſquares of AC, CD; but the ſquare of AF is equal (by 47. 1.) to the ſquares of AE, EF; for the angle AEF is a right angle; therefore the ſquare of AF is double of the *ſquares* of AC, CD: but the *ſquares* of AD, DF are equal (by 47. 1.) to the *ſquare* of AF, for the angle at D is a right angle; therefore the *ſquares* of AD, DF are double of the ſquares of AC, CD: but DF is equal to DB; wherefore the ſquares of AD, DB are double of the ſquares of AC, CD.

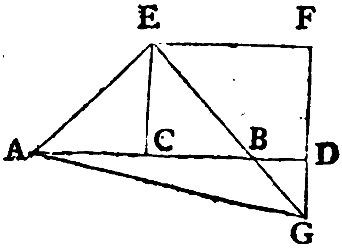
Wherefore if a ſtraight line be cut into equal and unequal *ſegments*; the ſquares of the unequal ſegments of the whole *line*, is double both of the ſquare of half *the line*, and of the *line* between the *points* of ſection. Which was to be demonſtrated.

PROP. X.

If a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the ſquare of the whole with the part produced, and the ſquare of the part produced, theſe *ſquares* taken both together are double both of the ſquare of half the line, and of the ſquare deſcribed upon the compounded *line*, that is of half *the line* and the part produced, as upon one *line*.

For let any ſtraight line AB be cut in halves at the *point* C; and let any ſtraight line BD be added to it in a ſtraight line; I ſay that the ſquares of AD, DB are double of the ſquares of AC, CD.

For from the point C let CE be drawn at right angles to AB (by 11. 1.) and let it be made equal to either of the lines AC, CB; and let AE, EB be joined; and through the *point* E let EF be drawn (by 31. 1.) parallel to AD; and through the *point* D let DF be drawn parallel to CE: And becauſe a certain ſtraight line EF hath fallen upon the parallel ſtraight lines EC, FD; (by 29. 1.) the angles CEF, EFD are therefore equal to two right angles; therefore the angles FEB, EFD are leſs than two right angles; but (by com. not. 11.) ſtraight lines produced from *angles* leſs than two right angles meet *one another*; therefore EB, FD being produced towards the parts B, D will meet *one another*; let them be produced and meet at the *point* G; and let AG be joined.



And becauſe AC is equal to CE, (by 5. 1.) the angle AEC is equal to the *angle* EAC; and the *angle* at C is a right angle; wherefore each of the angles EAC, AEC is half a right angle. Certainly for the ſame reaſon alſo, each of the angles CEB, EBC is half a right angle; wherefore the angle AEB is a right angle; and ſince EBC is half a right angle; therefore alſo (by 15. 1.) DBG is half a right angle; but (by 29. 1.) BDG is a right angle, for it is equal to DCE, for *they are* alternate; wherefore the remaining angle DGB is (by 32. 1.) half a right angle; therefore the angle DGB is equal to the angle DBG; ſo that alſo (by 6. 1.) the ſide BD is equal to the ſide GD: Again becauſe EGF is half a right angle; and the angle at F (by 34. 1.) is a right angle, for it is equal to the oppoſite angle at C: wherefore the remaining angle FEG is half a right angle; wherefore the angle EGF is equal to the angle FEG; ſo that (by 6. 1.) the ſide GF is equal to the ſide EF. And becauſe EC is equal to CA; alſo the ſquare of EC is equal to the ſquare of AC; therefore the ſquares of EC, CA are double of the ſquare of CA; but (by 47. 1.) the *ſquare* of EA is equal to the *ſquares* of EC, CA; therefore the ſquare of EA is double of the ſquare of AC. Again, becauſe GF is equal to EF, alſo the ſquare of FG is equal to the ſquare of EF; therefore the *ſquares* of FG, FE are double of the *ſquare* of FE; but the ſquare of EG is equal to the ſquares of EF, FG; wherefore the *ſquare* of EG is double of the ſquare of EF; but EF is (by 34. 1.) equal to CD; therefore the ſquare of EG is double of the *ſquare* of CD; but the *ſquare* of EA has been alſo demonſtrated to be double of the *ſquare* of AC; wherefore the ſquares of AE, EG are double of the ſquares of AC, CD: but the ſquare of AG is (by 47. 1.) equal to the ſquares of AE, EG; therefore the *ſquare* of AG is double of the *ſquares* of AC, CD; but the *ſquare* of AG is equal to the *ſquares* of AD, DG; wherefore the ſquares of AD, DG are double of the ſquares of AC, CD: but DG is equal to DB; therefore the ſquares of AD, DB are double of the ſquares of AC, CD.

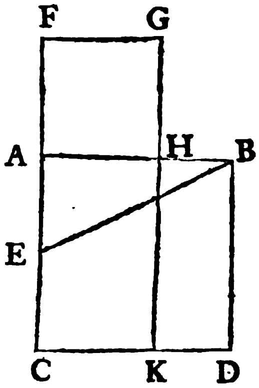
Wherefore if a ſtraight line be cut in halves, and any ſtraight line be added to it in a ſtraight line; the ſquare of the whole with the part produced, and the ſquare of the part produced, theſe *ſquares* taken both together are double both of the ſquare of half the line, and of the ſquare deſcribed upon the compounded *line*; that is of half *the line* and the part produced, as upon one *line*. Which was to be demonſtrated.

PROP. XI.

To cut a given ſtraight line, ſo that the rectangle contained by the whole and one of the ſegments may be equal to the ſquare of the remaining ſegment.

Let AB be the given ſtraight line; it is required to cut the *line* AB, ſo that the rectangle contained by the whole and one of the ſegments, may be equal to the ſquare of the remaining ſegment.

For let the ſquare ABDC be deſcribed upon AB; and let AC be cut in halves in the point E; and let BE be joined; and let CA be produced to F; and let EF be made equal to BE; and let the ſquare HF be deſcribed upon AF; and let GH be produced to K; I ſay that AB is cut in the point H ſo that the rectangle contained by AB, BH is equal to the ſquare of AH.



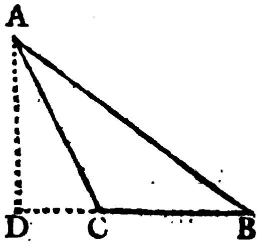
For becauſe the ſtraight line AC is cut in halves in the *point* E, and AF is added to it; therefore the rectangle contained by CF, FA together with the ſquare of AE is equal to the ſquare of EF; but EF is equal to EB; wherefore the rectangle contained by CF, FA together with the ſquare of AE is equal to the ſquare of EB; but (by 47.1.) the *ſquares* of BA, AE are equal to the *ſquare* of EB; for the angle at A is a right angle; therefore the *rectangle contained* by CF, FA together with the ſquare of AE is equal to the *ſquares* of BA, AE; let the common ſquare of AE be taken away; therefore the remainder, the *rectangle contained* by CF, FA is equal to the ſquare of AB; and the *rectangle* FK is the *rectangle contained* by CF, FA; for FA is equal to FG; and AD *is the ſquare* of AB; therefore FK is equal to AD; let AK *which is* common be taken away; therefore the remainder FH is equal to HD; and HD is the rectangle contained by AB, BH; for AB is equal to BD: and FH is the ſquare of AH; wherefore the rectangle contained by AB, BH is equal to the ſquare of AH.

Wherefore the given ſtraight line AB hath been cut in the *point* H, ſo that the rectangle contained by AB, BH is equal to the ſquare of HA. Which was to be done.

PROP. XII.

In obtuſe angled triangles, the ſquare, of the ſide ſubtending the obtuſe angle, is greater than the ſquares of the ſides containing the obtuſe angle by the *rectangle* twice *taken* contained by one of the *ſides* about the obtuſe angle upon which produced the perpendicular falls, and the *line* intercepted without, from the perpendicular to the obtuſe angle.

Let ABC be an obtuſe angled triangle, having an obtuſe angle, the *angle* ACB; and let the perpendicular AD be drawn from the point A, upon BC produced: I ſay that the ſquare of AB is greater than the ſquares of AC, CB by the rectangle contained by BC, CD *taken* twice.



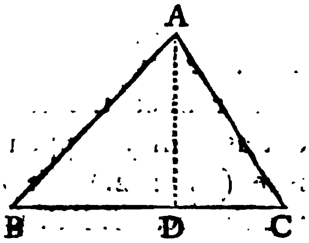
For ſince the ſtraight line BD hath been cut as it may happen in the point C; wherefore (by 4. 2.) the *ſquare* of BD is equal to the ſquares of BC, CD and the rectangle contained by BC, CD *taken* twice: let the common ſquare of AD be added; therefore the *ſquares* of BD, DA are equal to the ſquares of BC, CD, DA and to the rectangle contained by BC, CD *taken* twice: but the *ſquare* of AB is (by 47. 1.) equal to the *ſquares* of BD, DA; for the angle at D is a right angle; and the *ſquare* of AC is equal to the *ſquares* of CD, DA; wherefore the *ſquare* of AB is equal to the ſquares of BC, CA and to the rectangle contained by BC, CD *taken* twice; ſo that the ſquare of AB is greater than the ſquares of BC, CA by the rectangle contained by BC, CD *taken* twice.

Wherefore in obtuſe angled triangles the ſquare of the ſide ſubtending the obtuſe angle is greater than the ſquares of the ſides containing the obtuſe angle by the *rectangle* twice *taken*, contained by one of *the ſides* about the obtuſe angle upon which produced the perpendicular falls, and the *line* intercepted without from the perpendicular to the obtuſe angle. Which was to be demonſtrated.

PROP. XIII.

In acute angled triangles, the ſquare of the ſide ſubtending the acute angle is leſs than the ſquares of the ſides containing the acute angle, by the rectangle *taken* twice, contained by one of the *ſides* about the acute angle upon which the perpendicular falls and the *line* intercepted within, from the perpendicular to the acute angle.

Let there be an acute angled triangle, the *triangle* ABC, having the *angle* at B an acute angle; and let the perpendicular AD be drawn from the point A upon BC: I ſay that the ſquare of AC is leſs than the ſquares of CB, BA by the rectangle CB, BD *taken* twice.



For becauſe the ſtraight line CB hath been cut as it may happen in the *point* D; therefore the ſquares of CB, BD are equal to the rectangle CB, BD *taken* twice and to the ſquare of DC (by 7.2.): let the common ſquare of AD be added: therefore the ſquares of CB, BD, DA are equal to the rectangle contained by CB, BD *taken* twice, and to the ſquares of CD, DA: but the *ſquare* of AB is equal to the *ſquares* of BD, DA (by 47.1.) for the angle at D is a right angle; and the *ſquare* of AC is equal to the *ſquares* of CD, DA: wherefore the *ſquares* of CB, BA are equal to the rectangle contained by CB, BD, *taken* twice and the ſquare of AC; ſo that the *ſquare* of AC alone is leſs than the ſquares of CB, BA by the rectangle contained by CB, BD *taken* twice.

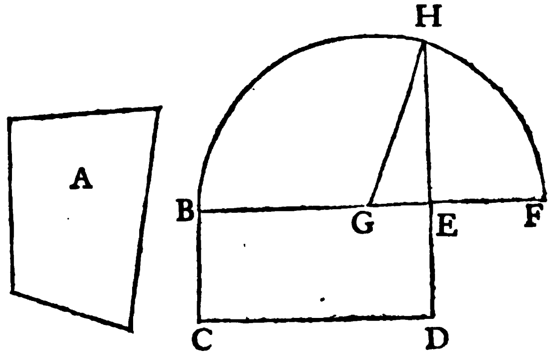
Wherefore in acute angled triangles, the ſquare of the ſide ſubtending the acute angle is leſs than the ſquares of the ſides containing the acute angle, by the rectangle *taken* twice, contained by one of the *ſides* about the acute angle upon which the perpendicular falls, and the line intercepted within, from the perpendicular to the acute angle. Which was to be demonſtrated.

PROP. XIV.

To make a ſquare equal to a given rectilineal figure.

Let A be the given rectilineal *figure*; it is required to make a ſquare equal to the given rectilineal *figure* A.

Let the right angled parallelogram BD be made (by 45. 1.) equal to the rectilineal *figure* A; if therefore BE be equal to ED what was required hath been done; for the ſquare BD is made equal to the rectilineal *figure* A: but if not, one of them BE, ED is greater: let BE be the greater, and let it be produced to F, and make EF equal to ED; and let FB be cut in halves (by 10. 1.) in the *point* G; and with the center G and at the diſtance of one of the lines GB, GF let the ſemicircle BHF be deſcribed, and let DE be produced to H, and let GH be joined.



Therefore becauſe the ſtraight line EF hath been cut into equal *ſegments* at the point G, and into unequal *ſegments* at E; therefore (by 5. 2.) the rectangle contained by BE, EF together with the ſquare of EG is equal to the ſquare of GF; but GF is equal to GH; therefore the *rectangle contained* by BE, EF together with the *ſquare* of EG is equal to the ſquare of GH; but the ſquares of HE, EG (by 47. 1.) are equal to the ſquare of GH; wherefore the *rectangle contained* by BE, EF together with the *ſquare* of GE is equal to the *ſquares* of HE, EG: let the common ſquare of EG be taken away; therefore what remains, the rectangle contained by BE, EF is equal to the ſquare of EH : but the rectangle contained by BE, EF is BD, for EF is equal to ED : therefore the parallelogram BD is equal to the ſquare of HE ; but BD is equal to the rectilineal *figure* A : therefore the rectilineal *figure* A is equal to the ſquare deſcribed upon EH.

Wherefore a ſquare is made equal to the given rectilineal *figure* A, the ſquare deſcribed upon EH. Which was to be done.

THE

ELEMENTS

OF

EUCLID.

BOOK III.

DEFINITIONS.

1. EQUAL circles, are thoſe of which the diameters are equal; or of which the *ſtraight lines* from the centers are equal.

2. A ſtraight line is ſaid to touch a circle, which meeting the circle, and being produced does not cut the circle.

3. Circles are ſaid to touch one another, which, meeting each other, do not cut one another.

4. In a circle ſtraight lines are ſaid to be equally diſtant from the center, when the perpendiculars drawn from the center upon them are equal. 5. But *that line* is ſaid to be more diſtant upon which the greater perpendicular falls.

6. A ſegment of a circle is the figure bounded by a ſtraight line, and the circumference of a circle. 7. But an angle of a ſegment is that contained by a ſtraight line and the circumference of a circle.

8. But an angle in a ſegment is; when any point is taken in the circumference of the ſegment, and from it ſtraight lines are joined to the extremities of the ſtraight line which is the baſe of the ſegment; the angle contained by the ſtraight lines ſo joined. 9. But when the ſtraight lines containing the angle receive any circumference, the angle is ſaid to ſtand upon that. 10. And a ſector of a circle is, when an angle ſtands at the center of the circle, the figure bounded by the ſtraight lines containing the angle, and the circumference intercepted by them.

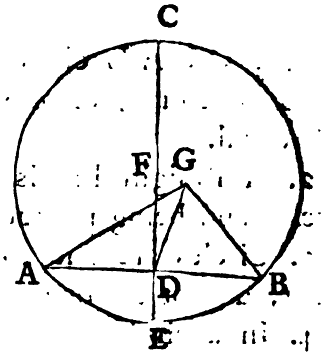
11. Similar ſegments of a circle are ſuch as receive equal angles; or in which the angles are equal to one another.

PROP. I.

To find the center of a given circle.

Let the given circle be *the circle* ABC; it is required to find the center of the circle.

Let any ſtraight line AB be drawn in it as it may happen; and let it be cut in halves at the *point* D, and from the point D let DC be drawn at right angles to AB, and let it be produced to E; and let CE be cut in halves at the *point* F: I ſay that F is the center of the circle ABC.



For if not, but if poſſible let it be G; and let GA, GD, GB be joined; and becauſe AD is (by conſt.) equal to DB, and DG common; certainly the two AD, DG are equal to the two GD, DB each to each; and the baſe GA is (by def. 15. 1.) equal to the baſe GB; for they are from the center G; wherefore the angle ADG is equal to the angle GDB; but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another, each of the equal angles is a right angle (by def. 10. 1.); therefore GDB is a right angle; but FDB is alſo a right angle; wherefore FDB is equal to GDB, the greater to the leſs, which is impoſſible; therefore the *point* G is not the center of the circle ABC : certainly in the ſame manner we ſhall demonſtrate that neither is any other but the *point* F.

Wherefore the point F is the center of the circle. Which was to be done.

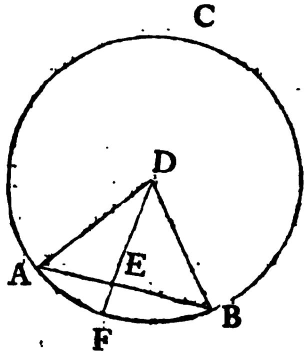
Cor. Certainly from this it is manifeſt, that if in a circle any ſtraight line, cut any ſtraight line in halves and at right angles, the center of the circle is in the cutting *line*.

PROP. II.

If two accidental points be taken in the circumference of a circle ; the ſtraight line joining theſe points will fall within the circle.

Let ABC be a circle, and let two accidental points be taken in the circumference of it, *as the points* A, B ; I ſay that the ſtraight line, joining A and B, will fall within the circle.

For if not, but if poſſible let AEB fall without ; and let the center of the circle ABC be taken (by 1. 3.), and let it be D ; and let AD, DB be joined ; and let DF be produced to E.



And ſince DA is equal to DB, therefore the angle DAE (by 5. 1.) is equal to DBE ; and becauſe one ſide of the triangle DAE is produced the ſide AEB, therefore (by 16. 1.) the angle DEB is greater than DAE : but DAE is equal to DBE ; therefore DEB is greater than DBE ; but (by 19. 1.) the greater ſide is extended under the greater angle ; therefore DB is greater than DE ; but DB is equal to DF ; therefore DF is greater than DE ; the leſs than the greater, *ſuppoſing E without the circle*, which is impoſſible : therefore the ſtraight line joining A and B will not fall without the circle : certainly in the ſame manner we ſhall demonſtrate that neither will it fall in the circumference : therefore within *the circle*.

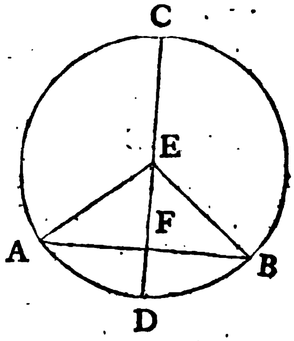
Wherefore if two accidental points be taken in the circumference of a circle, the ſtraight line, joining theſe points, will fall within the circle. Which was to be demonſtrated.

PROP. III.

If in a circle any ſtraight line through the center cut in halves any ſtraight line not through the center, it will alſo cut it at right angles; and if it cut it at right angles it will alſo cut it in halves.

Let ABC be a circle, and let CD any ſtraight line in it through the center cut in halves AB any ſtraight line, not through the center, in the point F; I ſay that it alſo cuts it at right angles.

For let the center of the circle ABC be taken (by 1. 3.); and let it be E, and let EA, EB be joined.



And becauſe AF is equal to FB, and FE common; certainly the two are equal to the two; and the baſe EA is equal to the baſe EB; and (by 8. 1.) the angle AFE is equal to the angle BFE: but when a ſtraight line ſtanding upon a ſtraight line makes the adjacent angles equal to one another, each of the equal angles is a right angle; therefore each of the angles AFE, BFE is a right angle: therefore CD through the center cutting in halves AB not paſſing through the center, alſo cuts it at right angles.

But let CD cut AB at right angles; I ſay that it alſo cuts it in halves; that is, that AF is equal to FB.

For the ſame things being conſtructed, becauſe EA from the center is equal to EB, the angle alſo EAF is equal (by 5. 1.) to EBF; but the right angle AFE is equal to the right angle BFE; therefore there are two triangles EAF, EBF having the two angles equal to the two angles, and one ſide equal to one ſide, viz. EF common to them, extended under one of the equal angles; they will therefore alſo have (by 26. 1.) the remaining ſides equal to the remaining ſides; wherefore AF is equal to BF.

Wherefore if in a circle, any ſtraight line through the center cut in halves any ſtraight line not through the center; it will alſo cut it at right angles; and if it cut it at right angles it will alſo cut it in halves. Which was to be demonſtrated.

PROP.

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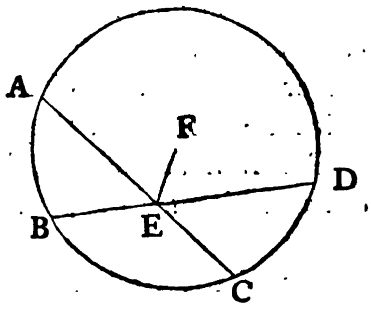
PROP. IV.

If in a circle two ſtraight lines cut one another, not paſſing through the center; they do not cut one another in halves.

Let ABCD be a circle, and let two ſtraight lines in it AC, BD cut one another in the point E, not paſſing through the center; I ſay that they do not cut one another in halves.

For if *it be* poſſible, let them cut one another in halves, ſo that AE is equal to EC and BE to ED; and let the center of the circle ABCD be taken (by 1. 3.), and let it be F; and let FE be joined.



Wherefore ſince a certain ſtraight line FE through the center cuts in halves a certain ſtraight line AC not drawn through the center, it will alſo cut it (by 3.3.) at right angles; therefore FEA is a right angle: Again becauſe a certain ſtraight line FE cuts in halves a certain ſtraight line BD, not *paſſing* through the center, it will alſo (by 3. 3.) cut it at right angles; wherefore FEB is a right angle; but FEA has been alſo demonſtrated to be a right angle: wherefore the *angle* FEA is equal to the *angle* FEB; the leſs to the greater which is impoſſible: wherefore the ſtraight lines AC, BD do not cut one another in halves.

Wherefore if in a circle two ſtraight lines cut one another, not paſſing through the center; they do not cut one another in halves. Which was to be demonſtrated.

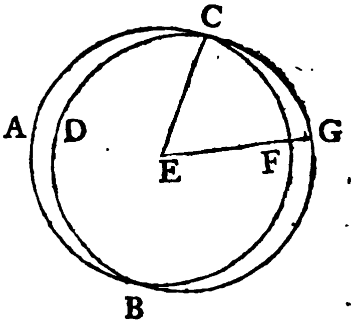
PROP. V.

If two circles cut one another, they will not have the ſame center.

For let two circles ABC, CDG cut one another in the points, B, C; I ſay they will not have the ſame center.

For if *it be* poſſible, let it be E; and let EC be joined; and let EFG be drawn as it may happen.

And becauſe the point E is the center of the circle ABC, EC is equal to EF: again becauſe the point E is the center of the circle CDG, EC is equal to EG; but EC has been already demonſtrated *to be* equal to EF: wherefore EF is equal to EG, the leſs to the greater, which is impoſſible: wherefore the point E is not the center of the circles ABC, CDG.



Therefore if two circles cut one another, they will not have the ſame center. Which was to be demonſtrated.

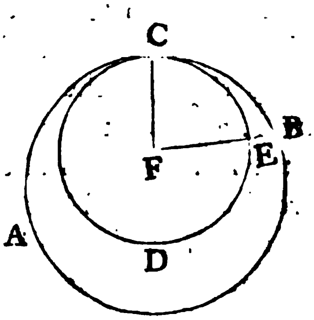
PROP. VI.

If two circles touch one another within, they will not have the ſame center.

For let the two circles ABC, CDE touch one another in the point C; I ſay that they will not have the ſame center.

For if *it be* poſſible, let it be F, and join FC, and let FEB be drawn as it may happen.

Wherefore ſince the point F is the center of the circle ABC; FC is equal to FB; again, becauſe the point F is the center of the circle CDE; FC is equal to FE; but FC has been alſo demonſtrated *to be* equal to FB; therefore FE is equal to FB, the leſs to the greater, which is impoſſible: wherefore the point F is not the center of the circles ABC, CDE.

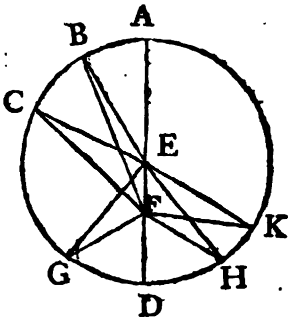


Wherefore if two circles touch one another within, they will not have the ſame center. Which was to be demonſtrated.

PROP. VII.

If any point be taken in the diameter of a circle, which is not the center of the circle; and from this point let certain ſtraight lines fall upon the circle, that will be the greateſt in which the center *is*; and the remainder *of the diameter* the leaſt; but of the others, the nearer to the line through the center is always greater than one more remote; and two equal ſtraight lines only will fall from the ſame point upon the circle, on each ſide of the leaſt.

Let ABCD be a circle, and let AD be its diameter; and let any point F be taken in AD, which is not the center of the circle; and let E be the center of the circle; and from the *point* F let certain ſtraight lines FB, FC, FG fall upon the circle ABCD: I ſay that FA is the greateſt, and FD the leaſt; but of the others FB is greater than FC; and FC than FG.



For let BE, CE, GE be joined.

And becauſe the two ſides of every triangle are greater than the remaining *ſide*; therefore EB, EF are greater than BF; but AE is equal to BE; wherefore BE, EF are equal to AF: therefore AF is greater than BF: Again, becauſe BE is equal to CE, and FE common; certainly the two BE, EF are equal to the two CE, EF; but the angle BEF is alſo greater than the angle CEF; wherefore (by 24. 1.) the baſe BF is greater than the baſe CF. Certainly for the ſame reaſon alſo CF is greater than FG.

Again, ſince GF, FE are (by 20. 1.) greater than EG; and EG is equal to ED; therefore GF, FE are greater than ED; Let EF *which is* common be taken away; therefore the remainder GF is greater than the remainder FD; wherefore FA is the greateſt, and FD the leaſt; and FB is greater than FC; and FC than FG.

I ſay that alſo only two equal ſtraight lines will fall from the point F upon the circle ABCD, *one* on each *ſide* of the leaſt FD: for let the *angle* FEH be made with the ſtraight line EF, and at the point E in it, equal to the angle GEF; and let FH be joined: wherefore becauſe GE is equal to EH, and EF common; certainly the two GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; wherefore (by 4. 1.) the baſe FG is equal to the baſe FH: I ſay that another *line* equal to FG will not fall upon the circle, from the point F: for if it be poſſible, let FK fall; and ſince FK is equal to FG, but FH is equal to FG; therefore FK is equal FH; the nearer to the *line* through the center equal to the *one* more remote; which is impoſſible.

Or

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OR THUS. Let EK be joined; and becauſe GE is equal to EK, and FE common, and the baſe GF equal to the baſe FK; therefore (by 8. 1.) the angle GEF is equal to the angle KEF; but the angle GEF is equal to the *angle* HEF; and therefore the *angle* HEF is equal to the *angle* KEF; the leſs to the greater which is impoſſible: therefore no other line will fall from the point F upon the circle equal to GF: therefore one only.

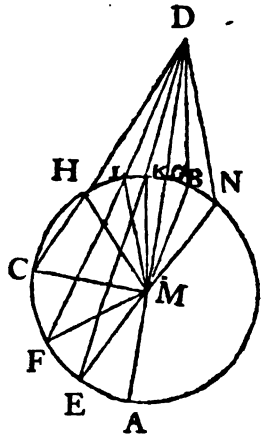
Wherefore if any point be taken in the diameter of a circle &c. Which was to be demonſtrated.

PROP. VIII.

If any point be taken without a circle, and certain ſtraight lines be drawn from the point to the circle, one of which *paſſes* through the center, but the other *lines* as it may happen; of thoſe ſtraight lines falling upon the concave circumference, the *one paſſing* through the center is the greateſt: but of the other *lines*, the *one* nearer the *line paſſing* through the center will always be greater than that more remote: But of the ſtraight lines falling upon the convex circumference, the leaſt is that between the point and the diameter: but of the other lines the nearer to the leaſt is leſs than that more remote; and two equal ſtraight lines only will fall upon the circle from the point, *one* on each *ſide* of the leaſt.

Let ABC be a circle; and let any point D be taken without the circle ABC: and let certain ſtraight lines be drawn from it to the circle viz. DA, DE, DF, DC; and let AD be through the center; I ſay that of the ſtraight lines falling upon AEFC the concave circumference, the greateſt is DA the *line* through the center: but the *line* nearer to the *one* through the center will be greater than *one* more remote; viz DE than DF and DF *greater* than DC; but of the ſtraight lines falling upon the convex circumference HLKG, DG is the leaſt, which is between the point D and the diameter AG; but the nearer to the leaſt DG is leſs than one more remote; viz. DK *leſs* than DL and DL than DH.

For let the center of the circle ABC be taken; and let it be M; and let ME, MF, MC, MK, ML, MH be joined.



And becauſe AM is equal to EM; let MD *which is* common be added; therefore AD is equal to EM, MD: but (by 20. 1.) EM, MD are greater than ED; and therefore AD is greater than ED: Again becauſe ME is equal to MF; and MD common; therefore EM, MD are equal to MF, MD; and the angle EMD is greater than the angle FMD; wherefore (by 24. 1.) the baſe ED is greater than the baſe FD: certainly in the ſame manner we ſhall demonſtrate alſo that DF is greater than CD: wherefore AD is the greateſt; and DE greater than DF; and DF than DC.

And ſince MK, KD are greater than MD; and MG equal to MK, therefore the remainder KD is greater than the remainder GD; ſo that GD is leſs than KD, wherefore it is the leaſt. And becauſe upon MD one of the ſides of the triangle MLD two ſtraight lines MK, KD are joined together within the triangle; therefore (by 21. 1.) MK, KD are leſs than ML, LD; of which MK is equal to ML; therefore the remainder DK is leſs than the remainder DL: certainly in the ſame manner we ſhall demonſtrate that DL is leſs than DH: wherefore DG is the leaſt; and DK is leſs than DL; and DL than DH.

Alſo I ſay that only two equal ſtraight lines will fall from the point D upon the circle; *one* on each ſide of DG the leaſt: with the ſtraight line MD, and at the point M in it, let the angle DMB be made equal to the angle KMD; and let DB be joined; and becauſe MK is equal to MB, and MD common; certainly the two KM, MD are equal to the two BM, MD, each to each; and the angle KMD is equal to the angle BMD: wherefore (by 4. 1.) the baſe DK is equal to the baſe DB: I ſay that another *ſtraight line* equal to the ſtraight line DK will not fall upon the circle from the point D: for if it be poſſible let it fall; and let it be DN; becauſe therefore DK is equal to DN; but DK is equal to DB; therefore alſo DB is equal to DN; the nearer to the leaſt DG equal to *one* more remote, which has been ſhewn to be impoſſible.

Or otherwise. Let MN be joined, and becauſe KM is equal to MN, and MD common, alſo the baſe DK equal to the baſe DN; wherefore the angle KMD is equal to the angle DMN; but the *angle* KMD is equal to BMD: therefore alſo BMD is equal to NMD; the leſs to the greater, which is impoſſible: Therefore more than two ſtraight lines will not fall upon the circle ABC from the point D, *one* on each *ſide* of the leaſt GD.

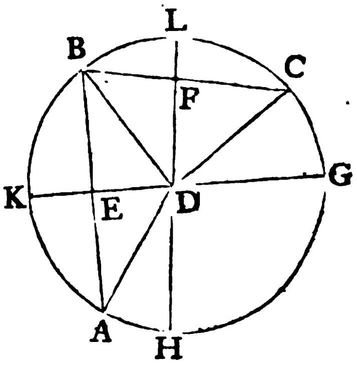
Wherefore if any point be taken without a circle &c. Which was to be demonſtrated.

PROP. IX.

If any point be taken within a circle, and more than two equal ſtraight lines fall from the point upon the circle: the point taken is the center of the circle.

Let ABC be a circle, and the point D within it; and let DA, DB, DC; more than two equal ſtraight lines, fall from the *point* D to the circle ABC: I ſay that the point D is the center of the circle ABC.

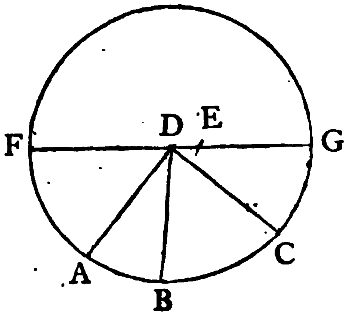
For let AB, BC be joined, and let them be cut in halves in the points E, F; ED, DF being joined; let them be produced to the points G, K, H, L.



Wherefore becauſe AE is equal to EB, and ED common: certainly the two AE, ED are equal to the two BE, ED; and the baſe DA is equal to the baſe DB; therefore (by 8. 1.) the angle AED is equal to the angle BED; wherefore each of the angles AED, BED is a right angle; wherefore GK cutting AB in halves cuts it alſo at right angles; and becauſe (by cor. prop. 1. 3.) if in a circle any ſtraight line cut any ſtraight line in halves and at right angles, the center of the circle is in the cutting line; therefore the center of the circle ABC is in GK: certainly for the ſame reaſon alſo the center of the circle ABC is in HL; and the ſtraight lines GK, HL have no other *point* in common, but the point D; therefore the point D is the center of the circle ABC.

Wherefore if any point be taken within a circle; and more than two equal ſtraight lines fall from the point upon the circle; the point taken is the center of the circle. Which was to be demonſtrated.

OTHERWISE. For let there be, any point D, taken within the circle ABC and let more than two equal ſtraight lines DA, DB, DC fall from the *point* D upon the circle ABC; I ſay that D the point taken is the center of the circle ABC.

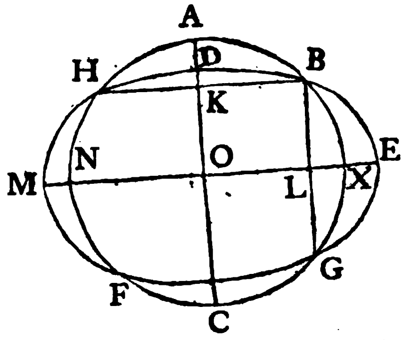


For if not; but if poſſible let it be E; and DE being joined, let it be produced to the points F, G; but FG is a diameter of the circle ABC: therefore becauſe a certain point hath been taken in FG the diameter of the circle ABC, which is not the center of the circle, viz. the *point* D; DG will be the greateſt (by prop. 7. 3.) and DC greater than DB; and DB than DA; but *they are* alſo equal; which is impoſſible: wherefore E is not the center of the circle ABC: certainly in the ſame manner we ſhall demonſtrate, that neither is any other but the *point* D: wherefore the point D is the center of the circle ABC.

PROP. X.

A circle does not cut a circle in more points than two.

For if *it be* poſſible; let the circle ABC cut the circle DEF in more points than two, viz. in B, G, H; and BG, BH being joined, let them be cut in halves in the points K, L; and from the points K, L having drawn KC, LM at right angles to BG, BH let them be produced to the points A, E.

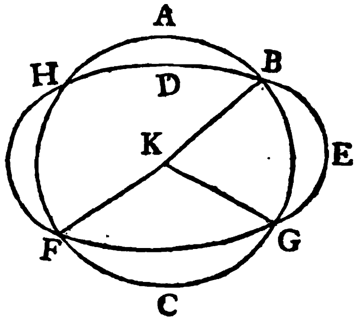


Wherefore becauſe in the circle ABC a certain ſtraight line AC cuts in halves a certain ſtraight line BH and at right angles; therefore the center of the circle ABC is in the ſtraight line AC: again, becauſe in the ſame circle ABC a certain ſtraight line NX cuts in halves and at right angles a certain ſtraight line BG;

therefore the center of the circle ABC is in NX; but it has alſo been demonſtrated to be in AC; and the ſtraight lines AC, NX meet one another in no *point* but the *point* O; therefore the point O is the center of the circle ABC: certainly in the ſame manner we ſhall demonſtrate that O is the center of the circle DEF; therefore two circles ABC, DEF cutting one another have the ſame center the *point* O which is impoſſible (by 5. 3.).

Wherefore a circle does not cut a circle in more points than two. Which was to be demonſtrated.

OTHERWISE. For again, let the circle ABC cut the circle DEF in more points than two; viz. in the *points* B, G, F; and let the center of the circle ABC be taken the *point* K; and let KF, KG, KB be joined.



Wherefore ſince a certain point hath been taken within the circle DEF, the point K; and from K more than two equal ſtraight lines KB, KF, KG have fallen upon the circle DEF: therefore (by 9. 3.) the point K is the center of the circle DEF; but K is alſo the center of the circle ABC; therefore two circles cutting one another have the ſame center K; which (by 5. 3.) is impoſſible.

Wherefore a circle does not cut a circle in more points than two. Which was to be demonſtrated.

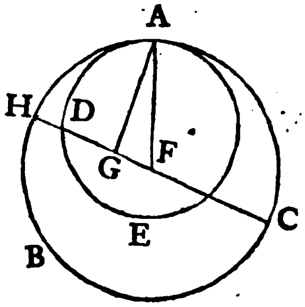
PROP. XI.

If two circles touch one another inwardly, and their centers be taken; the ſtraight line joining the centers of them, being produced, will fall upon the contact of the circles.

For let two circles ABC, ADE touch one another inwardly at the point A; and let F the center of the circle ABC be taken, and G the *center* of the *circle* ADE: I ſay that the ſtraight line joining G and F being produced will fall upon the point A.

For if not, but if poſſible let it fall as the *line* FGDH; and let AF, AG be joined.

Where-



Wherefore becauſe AG, GF are greater than FA, that is than FH; for FA is equal to FH for they are both from the center; let FG which is common be taken away; therefore the remainder AG is greater than the remainder GH; but AG is equal to GD (by def. 15. 1.); therefore GD is greater than GH; the leſs than the greater; which is impoſſible: wherefore the ſtraight line joining F and G will not fall without the contact at A; therefore upon it.

Wherefore if two circles touch one another inwardly, and their centers be taken; the ſtraight line joining the centers of them, being produced, will fall upon the contact of the circles. Which was to be demonſtrated.

OTHERWISE. But let it fall as GFC; and let GFC be produced in a ſtraight line to the point H; and let AG, AF be joined.

Wherefore ſince AG, GF are greater (by 20. 1.) than AF; but AF is equal to CF: that is to FH; let FG *which is* common be taken away; therefore the remainder AG is greater than the remainder GH; that is GD *is greater* than GH; the leſs than the greater which is impoſſible; in the ſame manner we ſhall ſhew the ſame *thing* to be abſurd, if the center of the greater circle be without the leſs.

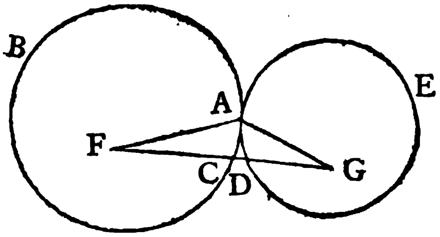
PROP. XII.

If two circles touch one another outwardly, the ſtraight line joining their centers will paſs through the contact.

For let two circles, ABC, ADE touch one another outwardly in the point A; and let F the center of the circle ABC be taken; and G the *center* of ADE; I ſay that the ſtraight line joining F and G will paſs through the contact A.

For if not, but if poſſible let it paſs as FCDG; and let AF, AG be joined.

Where-



Wherefore becauſe the point F is the center of the circle ABC; FA is equal to FC: again becauſe the point G is the center of the circle ADE; AG is equal to GD; but FA has been demonſtrated to be equal to FC; therefore FA, AG are equal to FC, DG; ſo that the whole FG is greater than FA, AG; but *it is* alſo leſs (by 20. 1.); which is impoſſible: wherefore the ſtraight line joining F and G will not not paſs through the contact at A; therefore *it will paſs* through it.

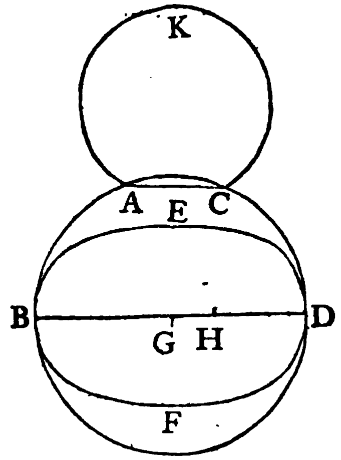
Wherefore if two circles touch one another outwardly, the ſtraight line joining their centers will paſs through the contact: Which was to be demonſtrated.

PROP. XIII.

A circle will not touch a circle in more points than one, whether it touch it within or without.

For if it be poſſible, let the circle ABDC firſt touch the circle EBFD within, in more points than one; viz. in the points, B, D.

And let G, the center of the circle ABDC be taken (by 1. 3.); and H the center of the circle EBFD.



Wherefore the ſtraight line joining G and H will (by 11. 3.) fall upon the *points* B, D: let it fall as BGHD; and ſince the point G is the center of the circle ABDC; BG is equal to GD; wherefore BG is greater than HD; therefore BH is much greater than HD; Again becauſe the point H is the center of the circle EBFD; BH is equal to HD; but it hath alſo been demonſtrated to be much greater than it; which is impoſſible; wherefore a circle does not touch a circle in more points than one inwardly.

I ſay that neither does it outwardly. For if it be poſſible; let the circle ACK touch the circle ABDC without in more points than one, in the points A, C and let AC be joined.

Wherefore becauſe any two points A, C have been taken in the circumference of each of the circles ABDC, ACK; the ſtraight line joining theſe points will fall (by 2. 3.) within both: but it hath fallen within the circle ABDC, and without the circle ACK; which is abſurd: wherefore a circle does not touch a circle outwardly in more points than one; but it has been demonſtrated that neither *does it* inwardly.

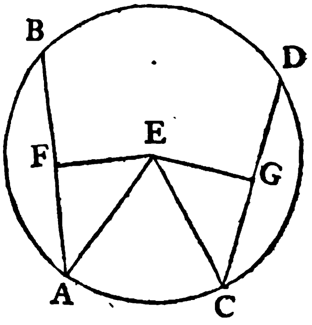
Wherefore a circle does not touch a circle in more points than one, whether it touch it on the inſide or on the outſide. Which was to be demonſtrated.

PROP. XIV.

The equal ſtraight lines, in a circle, are equally diſtant from the center; and the ſtraight lines being equally diſtant from the center are equal to one another.

Let ABDC be a circle; and let AB, CD be equal ſtraight lines in it; I ſay that they are equally diſtant from the center.

For let the center of the circle ABDC be taken (by 1. 3.); and let it be E; and let EF, EG be drawn perpendiculars from E to AB, CD; and let AE, EC be joined.



Wherefore ſince EF, a certain ſtraight line through the center, cuts a certain ſtraight line AB, not through the center, at right angles; it will cut it alſo in halves (by 3. 3.); wherefore AF is equal to FB: therefore AB is the double of AF. Certainly, for the ſame reaſon alſo, CD is double of CG; and AB is equal to CD; wherefore AF is equal to CG (by com. not. 7.): and becauſe AE is equal to EC; alſo the *ſquare* of AE *is* equal to the *ſquare* of EC: but the *ſquares* of AF, FE are equal (by 47. 1.) to the *ſquare* of AE; for the angle at F is a right angle: and the *ſquares* of EG, GC are equal to the *ſquare* of EC; for the angle at G is a right angle: wherefore the *ſquares* of AF, FE are equal to the *ſquares* of CG, GE; of which the *ſquare* of AF is equal to the *ſquare* of CG; for AF is equal to CG; therefore the remainder the *ſquare* of FE is equal to the remainder the *ſquare* of EG; wherefore FE is equal to EG: but in a circle, ſtraight lines are ſaid to be equally diſtant from the center, when perpendiculars drawn from the center upon them are equal (by def. 4. 3.) therefore AB, CD are equally diſtant from the center

But let the ſtraight lines AB, CD be equally diſtant from the center, that is, let FE be equal to EG: I ſay that AB is alſo equal to CD.

For the ſame things being conſtructed; certainly we ſhall demonſtrate, in the ſame manner, that AB is the double of AF; and CD of CG: aud becauſe AE is equal to EC; the *ſquare* of AE is alſo equal to the *ſquare* of EC; but the *ſquares* of EF, FA are equal to the *ſquare* of AE; and the *ſquares* of EG, GC are equal to the *ſquare* of CE; therefore the *ſquares* of EF, FA are equal to the *ſquares* of EG, GC; of which the ſquare of EG is equal to the ſquare of EF; therefore the remaining *ſquare* of AF is equal to the remaining *ſquare* of CG; wherefore AF is equal to CG; alſo AB is the double of AF, and CD the double of CG; wherefore AB is equal to CD.

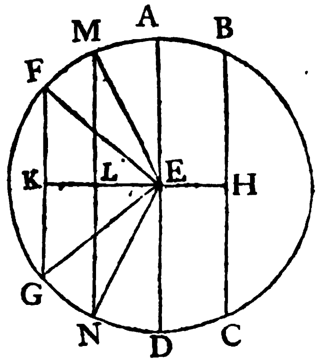
Wherefore the equal ſtraight lines, in a circle, are equally diſtant from the center; and the ſtraight lines, being equally diſtant from the center, are equal to one another. Which was to be demonſtrated.

PROP. XV.

The diameter is the greateſt *line* in a circle; but of the others, the *one* nearer to the center is greater than *any one* more remote.

Let ABCD be a circle; and let AD be a diameter of it; and E the center; and let BC be nearer the center E; but FG more remote; I ſay that AD is the greateſt, and BC greater than FG.

For let EH, EK be drawn perpendiculars from the center, to BC, FG; and becauſe BC is nearer to the center, but FG more remote; therefore (by def. 5. 3.) EK is greater than EH; let EL be made equal to EH; and through the point L, LM being drawn at right angles to EK let it be produced to N; and let EM, EN, EF, EG be joined.



And becauſe EH is equal to EL; (by 14. 3.) BC it equal to MN: Again ſince AE is equal to EM and ED to EN; therefore AD is equal to ME, EN; but (by 20. 1.) ME, EN are greater than MN; therefore alſo AD is greater than MN: but MN is equal to BC; therefore AD is greater than BC: And becauſe the two ME, EN are equal to the two FE, EG and the angle MEN is greater than the angle FEG; therefore (by 24. 1.) the baſe MN is greater than the baſe FG; but MN has been demonſtrated *to be* equal to BC; and BC is alſo greater than FG: wherefore the diameter AD *is* the greateſt; and BC greater than FG.

Wherefore the diameter is the greateſt *line* in a circle; but of the others the *one* nearer to the center is greater than *any one* more remote. Which was to be demonſtrated.

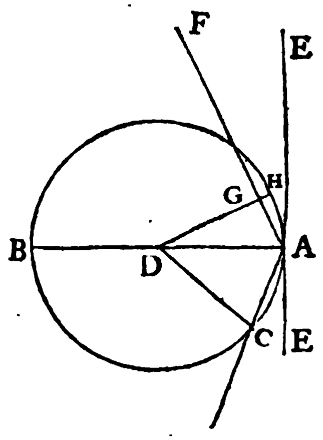
PROP. XVI.

The *ſtraight line* drawn at right angles to the diameter of a circle, from the extremity, will fall without the circle; and into the place between the ſtraight line and the circumference another ſtraight line will not fall; and the angle of the ſemicircle is greater than every acute rectilineal angle: but the remainder leſs.

Let ABC be a circle about the center D, and AB *its* diameter; I ſay that the *ſtraight line* drawn from its extremity, from the point A, at right angles to AB, will fall without the circle.

For if not, but if poſſible let it fall within, as AC and let DC be joined.

And becauſe DA is equal to DC, the angle DAC (by 5. 1.) is alſo equal to the angle ACD; but DAC is a right angle (by conſt.) wherefore alſo ACD is a right angle, therefore the angles DAC, ACD are equal to two right angles; which (by 17. 1.) is impoſ-



ſible; wherefore the *ſtraight line* drawn from the point A, at right angles to AB will not fall within the circle: certainly in like manner we ſhall demonſtrate that neither *will it fall* upon the circumference; therefore let it fall without as AE.

I ſay that another ſtraight line will not fall into the place, between the ſtraight line AE and the circumference HAC.

For if it be poſſible let it fall as FA; and let DG be drawn, from the point D, perpendicular to FA.

And becauſe AGD is a right angle, but DAG (by 17. 1.) leſs than a right angle; therefore (by 19. 1.) AD is greater than DG; but AD is equal to DH; therefore DH is greater than DG; the leſs than the greater, *ſuppoſing G without the circle,* which is impoſſible; wherefore another ſtraight line will not fall into the place between the ſtraight line and the circumference.

Alſo I ſay that the angle of the ſemicircle; the *angle* contained by the ſtraight line BA and the circumference HA; is greater than every acute rectilineal angle; but the remainder, the *angle* contained by the circumference HA and the ſtraight line AE is leſs than every acute rectilineal angle.

For if there be any rectilineal angle, greater than that contained by the ſtraight line BA and the circumference HA; but leſs than that contained by the circumference HA and the ſtraight line AE; a ſtraight line will fall into the place between the circumference HA, and the ſtraight line AE; which will make *an angle* contained by ſtraight lines greater, than that contained by the ſtraight line BA and the circumference HA; but leſs than that contained by the circumference HA and the ſtraight line AE: but it does not fall: therefore, the acute angle contained by the ſtraight lines will not be greater than the angle contained by the ſtraight line BA and the circumference HA, nor leſs than the *angle* contained by the circumference HA and the ſtraight line AE. Which was to be demonſtrated.

Cor.

Book III.

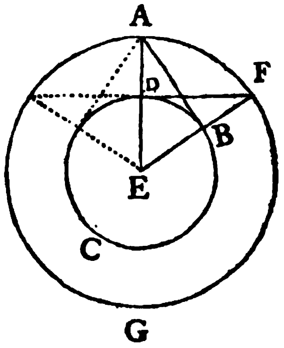
Cor. From theſe *demonſtrations* it is manifeſt that the *ſtraight line* drawn from the extremity at right angles to the diameter of the circle, touches the circle: and that a ſtraight line touches a circle in one point only: becauſe (by 2. 3.) the ſtraight line meeting it in two *points* has been ſhewn to fall within it.

PROP. XVII.

From a given point to draw a ſtraight line touching a given circle.

Let A be the given point, and BCD the given circle; it is required from the point A to draw a ſtraight line touching the circle BCD.

For let E the center of the circle be taken (by 1. 3.); and let AE be joined; and with the center E and at the diſtance EA let the circle AFG be deſcribed; and from the point D, let DF be drawn at right angles to AE; and let EBF, AB be joined: I ſay that from the point A the *ſtraight line* AB hath been drawn touching the circle BCD.



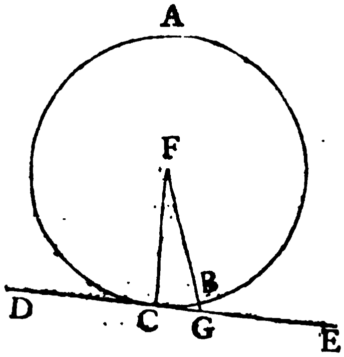
For ſince E is the center of the circles BCD, AFG; therefore EA is equal to EF; and ED to EB: certainly the two AE, EB are equal to the two FE, ED; and they contain a common angle, the *angle* at E; wherefore (by 4. 1.) the baſe DF is equal to the baſe AB and the triangle DEF is equal to the triangle EBA; and the remaining angles to the remaining angles; therefore the *angle* EBA is equal to EDF; but EDF is a right angle; wherefore EBA is a right angle; and EB is from the center: but the *ſtraight line* drawn from the extremity at right angles to the diameter of the circle touches the circle: wherefore AB touches the circle.

Wherefore from a given point, the *point* A, a ſtraight line AB hath been drawn touching the circle BCD. Which was to be done.

PROP. XVIII.

If any ſtraight line touch a circle, and any ſtraight line be drawn from the center to the *point of* contact; the *line* drawn will be perpendicular to the touching *line*.

Let any ſtraight line DE touch the circle ABC, in the point C; and let F the center of the circle ABC be taken; and from F to C let FC be drawn; I ſay that FC is perpendicular to DE.



For if not, let FG be drawn from the point F perpendicular to DE.

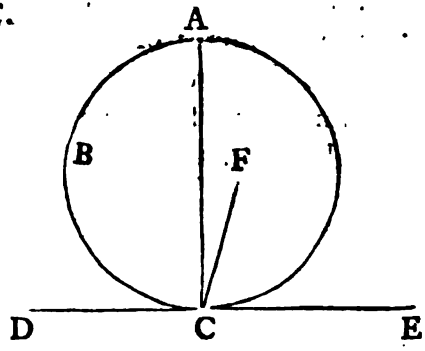
Wherefore becauſe FGC is a right angle, therefore GCF is an acute angle; but the greater ſide is extended under the greater angle; therefore FC *is* greater than FG; but FC *is* equal to FB: wherefore FB *is* greater than FG; the leſs than the greater which is impoſſible: therefore FG is not perpendicular to DE: certainly in the ſame manner we ſhall demonſtrate, that neither *is* any other but FC: wherefore FC is perpendicular to DE.

Wherefore if any ſtraight line touch a circle; and any ſtraight line be drawn from the center to the contact; the *line* drawn is perpendicular *to the tangent*. Which was to be demonſtrated.

PROP. XIX.

If any ſtraight line touch a circle, and a ſtraight line be drawn, from the contact, at right angles to the touching *line*; the center of the circle will be in the *line ſo* drawn.

For let any ſtraight line DE touch the circle ABC in the point C; and from C let CA be drawn at right angles to DE; I ſay that the center of the circle is in AC.



For if not, but if poſſible let it be F: and let CF be joined.

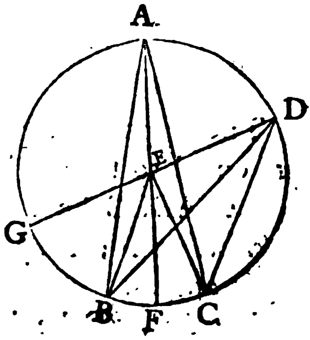
Wherefore becauſe a certain ſtraight line DE touches the circle ABC; and FC hath been drawn from the center to the contact; therefore FC is perpendicular to DE (by 18. 3.); therefore FCE is a right angle; but (by conſt.) ACE is a right angle; wherefore FCE is equal to ACE; the leſs to the greater; which is impoſſible; wherefore F is not the center of the circle ABC: certainly in the ſame manner we ſhall demonſtrate, that neither is it any other, but *ſome point* in AC.

Wherefore if any ſtraight line touch a circle, and a ſtraight line be drawn, from the contact, at right angles to the touching *line*; the center of the circle will be in the *line ſo* drawn. Which was to be demonſtrated.

PROP. XX.

In a circle, the angle at the center is double of the angle at the circumference; when the angles have the ſame circumference *for a* baſe.

Let ABC be a circle, and let BEC be an angle at the center of it; but BAC at the circumference *of it*; and let them have the ſame circumference BC *for a* baſe; I ſay that the angle BEC is double of the angle BAC.



For AE being joined let it be produced to F.

Wherefore becauſe EA is equal to EB; alſo (by 5. 1.) the angle EAB is equal to EBA; wherefore the angles EAB, EBA are double of EAB: but (by 32. 1.) BEF is equal to EAB, EBA; wherefore BEF is double of EAB: Certainly for the ſame reaſon FEC is double of EAC: therefore the whole BEC is double of the whole BAC.

Let it be bent in a different direction; and let the other angle be BDC; and DE being joined let it be produced to G: certainly in the ſame manner we ſhall demonſtrate that the angle GEC is double of the *angle* GDC; of which GEB is double of GDB; therefore the remainder BEC is double of the remainder BDC.

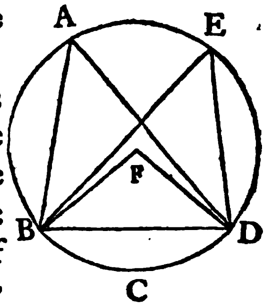
Wherefore in a circle, the angle at the center is double of the angle at the circumference; when the angles have the ſame circumference *for* a baſe. Which was to be demonſtrated.

PROP.

PROP. XXI.

In a circle, the angles in the ſame ſegment are equal to one another.

Let ABCD be a circle, and let the angles BAD, BED be in the ſegment BAED; I ſay that the angles BAD, BED are equal to one another.



For let the center of the circle ABCD be taken; and let it be F; and join BF, FD.

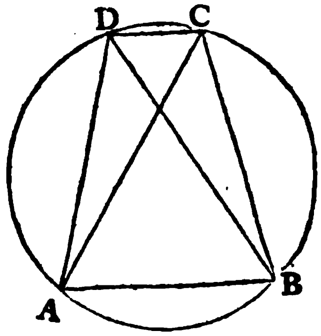
And becauſe the angle BFD is at the center; but BAD at the circumference, and have the ſame circumference BCD *for* a baſe; therefore the angle BFD is double of the angle BAD: certainly for the ſame reaſon BFD is double of BED: wherefore (by com. not. 7.) the *angle* BAD is equal to BED.

Wherefore in a circle, the angles in the ſame ſegment are equal to one another. Which was to be demonſtrated.

PROP. XXII.

The oppoſite angles, of quadrilateral *figures inſcribed* in circles, are equal to two right angles.

Let ABCD be a circle; and let ABCD be a quadrilateral *figure inſcribed* in it; I ſay that the oppoſite angles of it are equal to two right angles.



Let AC, BD be joined.

And becauſe (by 32. 1.) the three angles of every triangle are equal to two right angles; CAB, ABC, BCA the three angles of the triangle ABC are equal to two right angles: but CAB is equal to BDC (by 21. 3.); for they are in the ſame ſegment BADC: and ACB is equal to ADB (by 21. 3.): for they are in the ſame ſegment ADCB; wherefore the whole *angle* ADC is equal to the *two angles* BAC, ACB; let ABC *which is* common be added; therefore

fore the *three angles* ABC, BAC, ACB are equal to the *two* ABC, ADC: but the angles ABC, BAC, ACB are equal to two right angles; alſo the angles ABC, ADC are equal to two right angles: Certainly in the ſame manner we ſhall demonſtrate, that the angles BAD, DCB are equal to two right angles.

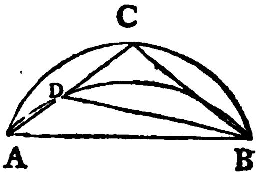
Wherefore the oppoſite angles, of quadrilateral *figures inſcribed* in circles, are equal to two right angles. Which was to be demonſtrated.

PROP. XXIII.

Two ſimilar and unequal ſegments of circles will not ſtand upon the ſame ſtraight line, towards the ſame parts.

For if it be poſſible; let ACB, ADB two ſimilar and unequal ſegments of circles ſtand upon the ſame ſtraight line AB, towards the ſame parts, and let ADC be drawn; and let CB, DB be joined.

Wherefore becauſe the ſegment ACB is ſimilar to the ſegment ADB: and ſimilar ſegments of circles are (by def. 11. 3.) thoſe receiving equal angles; therefore the angle ACB is equal to the *angle* ADB; the outward to the inward; which (by 16. 1.) is impoſſible.



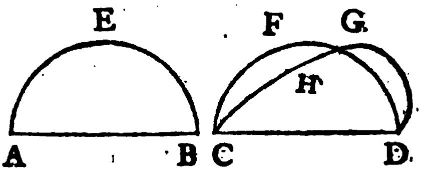
Wherefore two ſimilar and unequal ſegments of circles will not ſtand upon the ſame ſtraight line, towards the ſame parts. Which was to be demonſtrated.

PROP. XXIV.

The ſimilar ſegments of circles, upon equal ſtraight lines, are equal to one another.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; I ſay that the ſegment AEB is equal to the ſegment CFD.

For the ſegment AEB being applied to the *ſegment* CFD; and the point A being placed upon the *point* C; and the ſtraight line AB upon CD; the point B will apply itſelf to the point D; becauſe AB is equal to CD; but the ſtraight



ſtraight line AB applying itſelf to CD, the ſegment AEB (by 23. 3.) will apply itſelf to CFD: for if the ſtraight line AB ſhall apply itſelf to CD; but the ſegment AEB will not apply itſelf to CFD; but ſhall change its direction as CHGD: but a circle does not cut a circle in more points than two (by 10. 3.); but the *circle* CHGD cuts CFD in more points than two; viz. in C, G, D; which is impoſſible: wherefore the ſtraight line AB being applied to CD, alſo the ſegment AEB will not, not apply itſelf to CFD; therefore it will apply itſelf, and will be equal to it.

Wherefore the ſimilar ſegments of circles, upon equal ſtraight lines, are equal to one another, Which was to be demonſtrated.

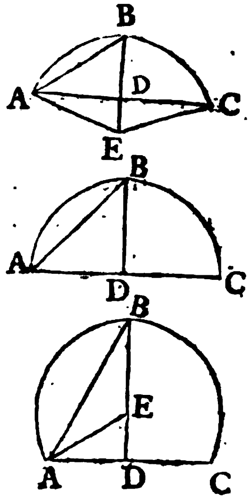
PROP. XXV.

A ſegment of a circle being given to deſcribe the circle of which it is a ſegment.

Let ABC be the given ſegment of the circle; it is required to deſcribe the circle, of which ABC is a ſegment.

For let AC be cut in halves (by 10. 1.) in the *point* D; and from the point D let DB be drawn at right angles to AC; and let AB be joined: therefore the angle ABD is either greater, equal or leſs than the *angle* BAD.

Firſt, let it be greater; and let the *angle* BAE be made, with the ſtraight line AB and at the point A in it, equal to the angle ABD; and let DB be produced to E; and let EC be joined.



Wherefore becauſe the angle ABE is equal to BAE; therefore (by 6. 1.) the ſtraight line BE is equal to EA: And becauſe AD is (by conſt.) equal to DC and DE common; certainly the two AD, DE are equal to the two CD, DE each to each; and the angle ADE is equal to the angle CDE; for each is a right angle; therefore alſo (by 4. 1.) the baſe AE is equal to the baſe EC; but AE has been demonſtrated *to be* equal to EB; therefore alſo BE is equal to CE; wherefore the three *ſtraight lines* AE, EB, EC are equal to one another: therefore a circle deſcribed with the center E and at the diſtance of one of the *lines* AE, EB, EC will alſo paſs through the remaining points, and will be the *circle* to be deſcribed (by 3. 3.): Wherefore the ſegment of a circle being given the circle has been deſcribed; and it is manifeſt that the ſegment ABC is leſs than a ſemicircle; on this account, becauſe E the center of it falls without.

In like manner, if the angle ABD is equal to the angle BAD; AD becoming equal to either of the *lines* BD, DC; therefore the three DA, DB, DC will be equal to one another; and D will be the center of the circle compleated; and certainly ABC will be a ſemicircle.

But if the *angle* ABD be leſs than BAD; alſo let us make with the ſtraight line AB, and at the point A in it, an angle equal to the *angle* ABD; the center will fall within the ſegment ABC in the *ſtraight line* DB; and certainly the ſegment ABC will be greater than a ſemicircle.

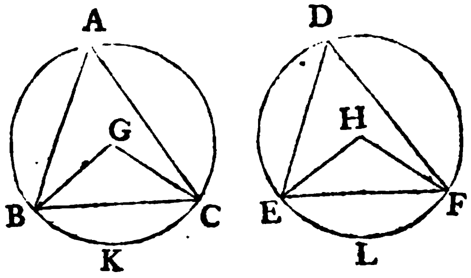
Wherefore a ſegment of a circle being given, the circle has been deſcribed, of which it is a ſegment. Which was to be done.

PROP. XXVI.

In equal circles, the equal angles ſtand upon equal circumferences; whether they ſtand at the center, or at the circumference.

Let ABC, DEF be equal circles; and let BGC, EHF be equal angles in them, at *their* centers; and BAC, EDF at the circumference: I ſay that the circumference BKC is equal to the circumference ELF.

For let BC, EF be joined.



And becauſe the circles ABC, DEF are equal; the ſtraight lines from their centers are equal (by def. 1. 3.): the two BG, GC are equal to the two EH, HF and (by ſupp.) the angle at G is equal to the *angle* at H: wherefore the baſe BC is equal to the baſe EF: and becauſe the angle at A is equal to the *angle* at D; therefore the ſegment BAC is ſimilar to the ſegment EDF (by def. 11. 3.); and they are upon equal ſtraight lines BC, EF; but ſimilar ſegments *being* upon equal ſtraight lines are equal (by 24. 3.); wherefore the ſegment BAC *is* equal to the ſegment EDF; but alſo the whole circle ABC is equal to the whole circle DEF; therefore the remaining ſegment BKC is equal to the remaining *ſegment* ELF; wherefore the circumference BKC is equal to the circumference ELF.

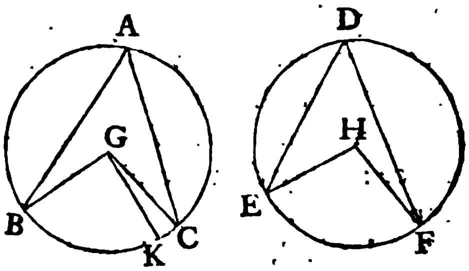
Wherefore in equal circles, the equal angles ſtand upon equal circumferences; whether they ſtand at the center or at the circumference. Which was to be demonſtrated.

PROP. XXVII.

In equal circles, the angles which ſtand upon equal circumferences are equal to one another; whether they ſtand at the center or at the circumference.

For let the angles BGC, EHF ſtand at the centers G, H in the equal circles ABC, DEF; and upon the equal circumferences BC, EF; and the angles BAC, EDF at the circumferences; I ſay that the angle *BGC* is equal to the *angle* EHF; and BAC to EDF.

If the *angle* BGC be equal to EHF; it is plain (by 20. 3.) that the *angle* BAC is equal to the *angle* EDF; but if *the angle BGC be* not *equal to the angle* EHF; one of them is greater; let BGC be the greater, and let the *angle* BGK (by 23. 1.) be made with the ſtraight line BG and at the point G in it, equal to the angle EHF: but (by 26. 3.) equal angles ſtand upon equal circumferences, when they are at the centers; wherefore the circumference BK *is* equal to the circumference EF; but EF (by ſupp.) is equal to BC; wherefore BK is equal to BC; the leſs to the greater; which is impoſſible: wherefore the angle BGC is not unequal to EHF; therefore equal; and the *angle* at A is half of the *angle* BGC; and the angle at D is half of the angle EHF: wherefore the angle at A is equal to the *angle* at D.



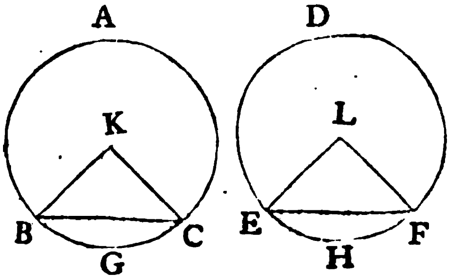
Wherefore in equal circles, the angles which ſtand upon equal circumferences are equal to one another; whether they ſtand at the center or at the circumference. Which was to be demonſtrated.

PROP. XXVIII.

Equal ſtraight lines in equal circles cut off equal circumferences; the greater *equal* to the greater; and the leſs *equal* to the leſs.

Let ABC, DEF be equal circles; and let BC, EF be equal ſtraight lines in them, cutting off the greater circumferences BAC, EDF; and BGC, EHF the leſs; I ſay that the greater circumference BAC is equal to the greater circumference EDF; and the leſſer circumference BGC *is* equal to the leſs EHF.

For let K, L the centers of the circles be taken; and let BK, KC, EL, LF be joined.



And becauſe the circles are equal; the lines alſo from their centers are eqnal (by def. 1.); therefore the two BK, KC are equal to the two EL, LF; and the baſe BC is (by ſupp.) equal to the baſe EF; wherefore (by 8. 1.) the angle BKC is equal to the angle ELF; but equal angles ſtand upon equal circumferences, when they are at the center (by 26. 3.); therefore the circumference BGC is equal to the circumference EHF; but the whole circle ABC is equal to the whole circle DEF; wherefore the remaining circumference BAC, is equal to the remaining circumference EDF.

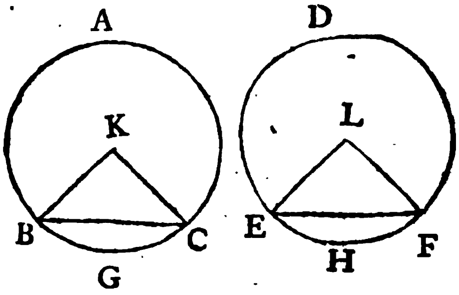
Wherefore equal ſtraight lines in equal circles cut off equal circumferences; the greater *equal* to the greater; and the leſs *equal* to the leſs. Which was to be demonſtrated.

PROP. XXIX.

In equal circles, equal ſtraight lines are extended under equal circumferences.

Let ABC, DEF be equal circles; and in them let the equal circumferences BGC, EHF be taken; and let the ſtraight lines BC, EF be drawn; I ſay that the ſtraight line BC is equal to EF.

For let K, L, the centers of the circles be taken; and let BK, KC, EL, LF be joined.



And becauſe the circumference BGC is equal to the circumference EHF; alſo (by 27. 3.) the angle BKC is equal to the *angle* ELF; and becauſe the circles ABC, DEF are equal; the lines from their centers are equal (by def. 1. 3.); therefore the two BK, KC are equal to the two EL, LF; and they contain equal angles; therefore (by 4. 1.) the baſe BC is equal to the baſe EF.

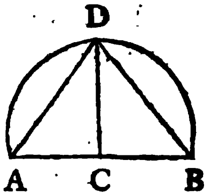
Wherefore, in equal circles, equal ſtraight lines are extended under equal circumferences. Which was to be demonſtrated.

PROP. XXX.

To cut a given circumference in halves.

Let ABD be the given circumference; it is required to cut the given circumference ADB in halves.

Let AB be joined, and let it be cut in halves (by 10. 1.) at the *point* C; and from the point C, let CD be drawn at right angles to the ſtraight line AB; and let AD, DB be joined.



And ſince AC is equal to CB; and CD common; certainly the two AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD; for each *of theſe is* a right angle; and (by 4. 1.) the baſe AD is equal to the baſe DB; but equal ſtraight lines cut off equal circumferences (by 28. 3.); the greater *equal* to the greater; and the leſs *equal* to the leſs: and each of the circumferences AD, DB is leſs than a ſemicircle; wherefore the circumference AD is equal to the circumference DB.

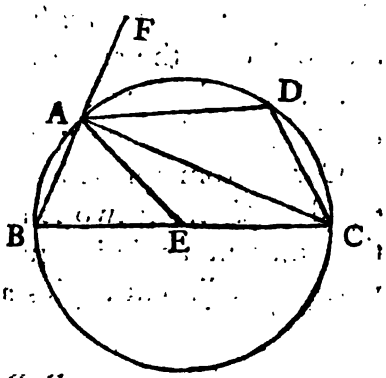
Wherefore the given circumference hath been cut in halves. Which was to be done.

PROP. XXXI.

In a circle, the angle in a ſemicircle is a right angle; but the *angle* in a greater ſegment is leſs than a right angle; and the *angle* in a ſegment leſs *than a ſemicircle is* greater than a right angle: and beſides the angle of a greater ſegment is greater than a right angle; but the angle of a ſegment leſs *than a ſemicircle is* leſs than a right angle.

Let ABCD be a circle; and let BC be the diameter of it, and E the center; alſo let BA, AC, AD, DC be joined; I ſay that the angle in the ſemicircle BAC is a right angle; but that the angle in the ſegment ABC greater than a ſemicircle, viz. the *angle* ABC, *is* leſs than a right angle; and that the angle in the ſegment ADC leſs than a ſemicircle is greater than a right angle.

Let AE be joined; and let BA be produced to F.



And becauſe BE is equal to EA, the angle EAB (by 5. 1.) is alſo equal to EBA: again becauſe EA is equal to EC, the angle EAC is alſo equal to ACE; wherefore the whole *angle* BAC is equal to the two *angles* ABC, ACB: but the *angle* FAC, the outward *angle* of the triangle ABC, is alſo equal to the two angles ABC, ACB (by 32. 1.): wherefore the angle BAC is equal to the *angle* FAC; therefore *each of them is* a right angle: wherefore the angle BAC, in the ſemicircle BAC is a right angle.

And ſince the two angles ABC, BAC of the triangle ABC are leſs than two right angles; and BAC is a right angle: therefore the angle ABC is leſs than a right angle; and it is in the ſegment ABC greater than a ſemicircle.

And becaeſe ABCD is a quadrilateral *figure inſcribed* in a circle; and the oppoſite angles of quadrilateral *figures* in circles are (by 22. 3.)

22. 3.) equal to two right angles; wherefore the angles ABC, ADC are equal to two right angles; but ABC is leſs than a right angle; therefore the remainder ADC is greater than a right angle; and it is in a ſegment leſs than a ſemicircle.

I ſay alſo that the angle of a greater ſegment, the angle contained by the circumference ABC and the ſtraight line AC, is greater than a right angle; but the angle of the ſegment leſs *than a ſemicircle*, the *angle* contained by the circumference ADC and the ſtraight line AC is leſs than a right angle: and it is manifeſt from hence. For becauſe the angle contained by the ſtraight lines BA, AC is a right angle; therefore the *angle* contained by the circumference ABC and the ſtraight line AC is greater than a right angle (by com. not, 9.): Again becauſe the *angle contained* by the ſtraight lines CA, AF is a right angle; therefore the *angle contained* by the ſtraight line AC and the circumference ADC is leſs than a right angle.

Otherwise. A demonſtration that BAC is a right angle. Becauſe the *angle* AEC (by 32. 1.) is double of the *angle* BAE; for it is equal to the two inward and oppoſite *angles*; and AEB is alſo double of the *angle* EAC; wherefore the *angles* AEB, AEC are double of the angle BAC; but the *angles* AEB, AEC are (by 13. 1.) equal to two right angles; wherefore the *angle* BAC is a right angle. Which was to be demonſtrated.

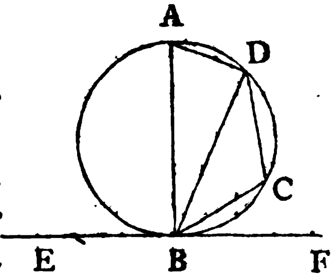
Cor From this it is manifeſt that if one angle of a triangle be equal to *the other* two it is a right angle: for this *reaſon* becauſe the adjacent angle is equal to the ſame two: but (by def. 10. 1.) when the adjacent angles are equal, they are right angles.

PROP. XXXII.

If any ſtraight line touch a circle; and *if* from the contact any ſtraight line be drawn to the circle, cutting the circle; the angles which it makes with the touching *line* will be equal to the angles in the alternate ſegments of the circle.

For let any ſtraight line EF touch the circle ABCD in the point B; and from the point B let BD any ſtraight line be drawn to the circle ABCD cutting it; I ſay that the angles which BD makes with the touching line EF will be equal to the angles in the alternate ſegments of the circle, that is, that the angle FBD is equal to the angle conſtituted in the ſegment DAB; and the angle EBD is equal to the *angle* in the ſegment DCB.

For let BA be drawn from the *point* B at right angles to EF; and let C any accidental point be taken in the circumference BD; and let AD, DC, CB be joined.



And becauſe a certain ſtraight line EF touches the circle ABCD in the point B; and from the contact at B the *ſtraight line* AB has been drawn at right angles to the touching *line*; the center of the circle ABCD is (by 19. 3.) in the *ſtraight line* AB: therefore the angle ADB, being in a ſemicircle, is (by 31. 3.) a right angle; therefore BAD, ABD, the remaining *angles*, are equal to one right angle: but ABF is alſo a right angle (by conſt.); wherefore the angle ABF is equal to the angles BAD, ABD; let the common angle ABD be taken away; therefore the remaining angle DBF is equal to the angle in the alternate ſegment of the circle, viz. the *angle* BAD: And becauſe ABCD is a quadrilateral *figure inſcribed* in a circle, its oppoſite angles are (by 22. 3.) equal to two right angles; therefore the *angles* DBF, DBE are equal to BAD, BCD; of which BAD has been demonſtrated to be equal to DBF; therefore the remaining *angle* DBE is equal to the *angle* in the alternate ſegment DCB of the circle, viz. the angle DCB.

Wherefore if any ſtraight line touch a circle; and if from the contact any ſtraight line be drawn to the circle, cutting the circle; the angles which it makes with the touching *line* will be equal to the angles in the alternate ſegments of the circle. Which was to be demonſtrated.

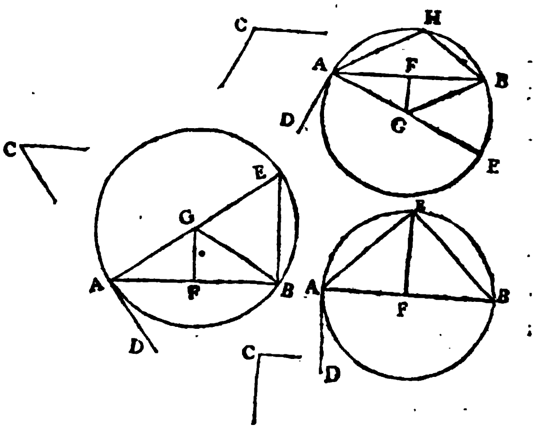
PROP. XXXIII.

Upon a given ſtraight line to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle.

Let the given ſtraight line be AB; and the *angle* at C, the given rectilineal angle: it is required, upon the given ſtraight line AB, to deſcribe a ſegment of a circle containing an angle equal to the *angle* at C : but the *angle* at C is either an acute *angle* or a right angle or an obtuſe *angle*.

Firſt let it be an acute *angle*, as in the firſt figure ; and let the *angle* BAD be made, with the ſtraight line AB and at the point A in it, equal to the angle at C ; wherefore alſo the angle BAD is acute : and let AE be drawn, from the point A, at right angles to AD ; and let AB be cut in halves at F ; and from the point F let FG be drawn at right angles to AB ; and let GB be joined.

And becauſe AF is equal to FB and FG common ; certainly the two AF, FG are equal to the two BF, FG ; and the angle AFG is (by conſt.) equal to BFG ; therefore (by 4. 1.) the baſe AG is equal to the baſe GB ; therefore the circle deſcribed with the center G and at the diſtance AG, will alſo paſs through B : let it be deſcribed ; and let it be ABE ; and let BE be joined. Wherefore becauſe from the extremity of the diameter AE ; from the *point* A ; AD is at right angles to AE ; therefore AD touches the circle (by cor. to 16. 3.). And ſince a certain ſtraight line AD touches the circle ABE ; and from the contact at A a certain ſtraight line AB hath been drawn to the circle ABE ; therefore the angle DAB is equal to the angle AEB in the alternate ſegment of the circle ; but the angle DAB is equal to the angle at C ; therefore the angle at C is equal to the angle AEB : wherefore upon the given ſtraight line AB, a ſegment of a circle hath been deſcribed viz. AEB, containing the angle AEB equal to the given angle at C.



But let the *angle* at C be a right angle ; and again let it be required to deſcribe upon AB a ſegment of a circle containing an angle equal to the right angle at C : Again let the *angle* BAD be made equal to the right angle at C, as it is in the ſecond figure ; and let AB be cut in halves in F ; and with the center F, and at the diſtance of either of *the lines* AF, FB let the circle AEB be deſcribed. Therefore the ſtraight line AD touches the circle ABE; becauſe the angle at A is a right angle; and the angle BAD is equal to the *angle* in the ſegment AEB; for it is alſo a right angle (by 31. 3.) being in a ſemicircle: but the *angle* BAD is equal to the *angle* at C: wherefore, again a ſegment of a circle viz. AEB has been deſcribed upon AB, containing an angle equal to the *angle* at C.

But let the *angle* at C be an obtuſe *angle*: and let the angle BAD be made equal to it, with the ſtraight line AB and at the point A, as it is in the third figure; and let AE be drawn at right angles to AD; and again let AB be cut in halves at F; and let FG be drawn at right angles to AB; and let GB be joined.

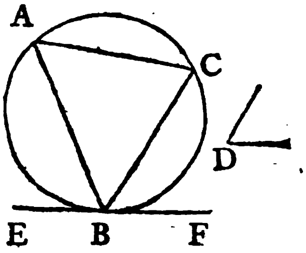
And again, becauſe AF is equal to FB and FG common; certainly the two AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG: therefore the baſe AG is equal to the baſe GB; wherefore a circle deſcribed with the center G; and at the diſtance AG, will alſo paſs through B; let it paſs as AEB: And becauſe AD has been drawn at right angles to the diameter AE from *its* extremity: therefore AD (by cor. 16. 3.) touches the circle AEB: and from the contact at A, the *ſtraight line* AB hath been drawn; wherefore the angle BAD is equal to AHB the angle contained in the alternate ſegment of the circle; but the angle BAD is equal to the *angle* at C: therefore alſo the angle in the ſegment AHB is equal to the *angle* at C. Wherefore upon the given ſtraight line AB, a ſegment of a circle; viz. AHB, has been deſcribed, containing an angle equal to the *angle* at C. Which was to be done.

PROP. XXXIV.

To cut off a ſegment from a given circle, containing an angle equal to a given rectilineal angle.

Let ABC be the given circle; and the given rectilineal angle, the *angle* at D: it is required to cut off a ſegment from the circle ABC, containing an angle equal to the angle at D.

Let EF be drawn touching the circle ABC in the point B; and let the angle FBC be made with the ſtraight line EF, and at the point B in it, equal to the angle at D.



Wherefore becauſe a certain ſtraight line EF touches the circle ABC, and from the contact at B, BC hath been drawn *cutting it*; therefore (by 32. 3.) FBC is equal to the angle contained in the alternate ſegment BAC: but the *angle* FBG is equal to the angle at D; wherefore alſo the angle in the ſegment BAC is equal to the angle at D.

Wherefore from a given circle ABC a ſegment BAC has been cut off, containing an angle equal to the given rectilineal angle at D. Which was to be done.

PROP. XXXV.

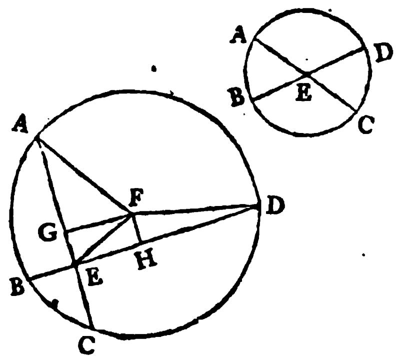
If in a circle two ſtraight lines cut one another, the rectangle contained by the ſegments of the one is equal to the rectangle contained by the ſegments of the other.

For in the circle ABCD let the two ſtraight lines AC, BD cut one another in the point E; I ſay that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

If AC, BD paſs through the center ſo that E be the center of the circle ABCD: it is manifeſt, AE, EC, DE, EB being equal, that the rectangle contained by AE, EC is equal to the rectangle contained by DE, EB.

But let AC, DB not paſs through the center; and let the center of the circle ABCD be taken (by 1. 3.); and let it be F; and from the *point* F, let FG, FH be drawn perpendiculars to AC, DB; and let FD, FA, FE be joined.

And becauſe GF a certain ſtraight line through the center cuts a certain ſtraight line AC not through the center at right angles; it will alſo cut it in halves (by 3. 3.); wherefore AG is equal to GC: wherefore becauſe the ſtraight line AC has been cut into equal *ſegments* at the *point* G; and into unequal *ſegments* at the *point* E; therefore (by 5. 2.) the rectangle contained by AE and



EC together with the ſquare of GE is equal to the ſquare of GA; let the *ſquare* of GF *which is* common be added: wherefore the *rectangle contained* by AE, EC together with the *ſquares* of GE, GF is equal to the *ſquares* of AG, GF: But the *ſquare* of FE is equal to the *ſquares* of EG, GF (by 47. 1.); and the *ſquare* of FA is equal to the *ſquares* of AG, GF: wherefore the rectangle contained by AE, EC together with the *ſquare* of FE is equal to the ſquare of FA: but FA is equal to FD; wherefore the *rectangle contained* by AE, EC together with the *ſquare* of FE is equal to the *ſquare* of FD. Certainly for the ſame reaſon alſo the *rectangle contained* by DE, EB together with the *ſquare* of FE is equal to the ſquare of FD: but it has alſo been demonſtrated that the *rectangle contained* by AE, EC together with the ſquare of FE is equal to the ſquare of FD: wherefore the rectangle contained by AE, EC together with the ſquare of FE is equal to the rectangle contained by DE, EB together with the ſquare of FE: let the ſquare of FE *which is* common be taken away; therefore the remaining rectangle contained by AE, EC is equal to the *remaining* rectangle contained by DE, EB.

Wherefore if in a circle two ſtraight lines cut one another, the rectangle contained by the ſegments of the one is equal to the rectangle contained by the ſegments of the other. Which was to be demonſtrated.

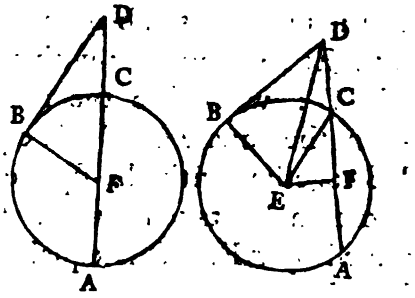
PROP. XXXVI.

If any point be taken without a circle, and two ſtraight lines fall from it upon the circle; and one of them cuts the circle and the other touches it; the rectangle contained by the whole cutting line and the ſegment without, taken between the point and the convex circumference, will be equal to the ſquare of the touching *line*.

For let D, any point without the circle ABC, be taken; and from the *point* D, let the two ſtraight lines DCA, DB fall upon

Book III. the circle ABC; and let DCA cut the circle ABC; and let DB touch *it*: I ſay that the rectangle contained by AD, DC is equal to the ſquare of DB: the *ſtraight line* DCA either paſſes through the center or not.

Firſt let it paſs through the center; and let F be the center of the circle ABC; and let FB be joined; therefore (by 18. 3.) FBD is a right angle: And becauſe the ſtraight line AC hath been cut in halves in the *point* F, and CD is added to it; therefore (by 6. 2.) the *rectangle contained* by AD, DC together with the *ſquare* of FC is equal to the *ſquare* of FD: but the *ſquare* of FD (by 47. 1.) is equal to the ſquares of FB, BD; for the angle FBD is a right angle: wherefore the rectangle contained by AD, DC together with the *ſquare* of FB is equal to the *ſquares* of FB, BD; let the common *ſquare* of FB be taken away; therefore the remaining *rectangle contained* by AD, DC is equal to the *remaining ſquare* of DB the touching *line*.



But let DA not paſs through the center of the circle ABC: and let E the center of the circle be taken; and from E let EF be drawn perpendicular to AC; and let EB, EC, BD be joined; wherefore EFD is a right angle; and becauſe EF a certain ſtraight line through the center, cuts at right angles AC a certain ſtraight line not through the center, it will alſo (by 3. 3.) cut it in halves; therefore AF is equal to FC: and becauſe the ſtraight line AC hath been cut in halves at F; and CD is added to it; therefore (by 6. 2.) the *rectangle contained* by AD, DC together with the *ſquare* of FC is equal to the *ſquare* of FD: let the common *ſquare* of FE be added; therefore the rectangle contained by AD, DC together with the ſquares of CF, FE are equal to the ſquares of DF, FE: but the ſquare of DE is equal to the ſquares of DF, FE (by 47. 1.); for EFD is a right angle; and the ſquare of CE is equal to the ſquares of CF, FE: Wherefore the *rectangle contained* by AD, DC together with the ſquare of EC is equal to the ſquare of ED: but CE is equal to EB; therefore the *rectangle contained* by AD, DC together with the ſquare of EB is equal to the ſquare of ED: but the ſquare of ED is equal to the ſquares of EB, BD; for the angle EBD is a right angle[b]: wherefore the rectangle contained by AD, DC together with the ſquare of EB is equal to the ſquares of EB, BD; let the common ſquare of EB be taken away; therefore the remaining *rectangle contained* by AD, DC is equal to the *remaining ſquare* of DB.

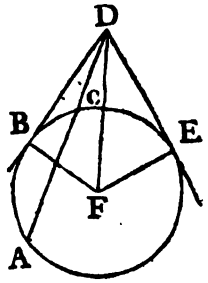
Wherefore if any point be taken without a circle, &c. Which was to be demonſtrated.

PROP. XXXVII.

If any point be taken without a circle, and two ſtraight lines fall from the point upon the circle, and one of them cuts the circle and the other meets *it*, and *if* the *rectangle contained* by the whole cutting line, and the *ſegment* without, taken between the point and the convex circumference, be equal to the *ſquare* of the line which meets it; the meeting *line* will touch the circle.

For let D be taken any point without the circle ABC; and let the two ſtraight lines DCA, DB fall from the point D, upon the circle ABC; and let DCA cut the circle, and DB meet *it*; and let the *rectangle contained* by AD, DC be equal to the *ſquare* of DB; I ſay that DB touches the circle ABC.

For let DE be drawn (by 17. 3.) touching the circle ABC; and let F the center of the circle ABC be taken (by 1. 3.); and let FE, FB, FD be joined: therefore the *angle* FED is (by 18. 3.) a right angle.



And ſince DE touches the circle ABC; and DCA cuts *it*; therefore (by 36. 3.) the *rectangle contained* by AD, DC is equal to the *ſquare* of DE; but the *rectangle contained* by AD, DC is ſuppoſed equal to the *ſquare* of DB; therefore the ſquare of DE is equal to the ſquare of DB; wherefore DE is equal to DB; and FE is alſo equal to FB; therefore the two DE, EF are equal to the two DB, BF; and DF is a common baſe to them; therefore (by 8. 1.) the angle DEF is equal to the angle DBF; but DEF is a right a right angle; therefore DBF is alſo a right angle; and FB produced is a diameter; but the *ſtraight line* drawn from the extremity at right angles to the diameter touches the circle ABC (by cor. to 16. 3.); certainly it will be demonſtrated in the ſame manner if the center happen to be in AC.

Wherefore if any point be taken without a circle, &c. Which was to be demonſtrated.

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THE

ELEMENTS

OF

EUCLID.

BOOK IV.

DEFINITIONS.

1. A Rectilineal figure is ſaid to be inſcribed in a rectilineal figure, when each of the angles of the inſcribed figure touches each ſide of the figure in which it is inſcribed. 2. And in like manner a figure is ſaid to be circumſcribed about a figure, when each ſide of the circumſcribed *figure* touches each angle of the figure about which it is circumſcribed. 3. And a rectilineal figure is ſaid to be inſcribed in a circle when each angle of the inſcribed *figure* touches the circumference of the circle. 4. And a rectilineal figure is ſaid to be circumſcribed about a circle, when each ſide of the circumſcribed figure (*is a tangent to the circle*) touches the circumference of the circle. 5. In like manner a circle is ſaid to be inſcribed in a figure, when the circumference of the circle touches each ſide of the *figure* within which it is inſcribed. 6. But a circle is ſaid to be circumſcribed about a figure, when the

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circumference of the circle touches each angle of the *figure* about which it is circumſcribed.

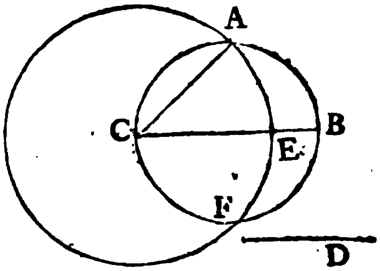
7. A ſtraight line is ſaid to be applied *or placed* in a circle, when the extremities of it are in the circumference of the circle.

PROP. I.

In a given circle to apply a ſtraight line, equal to a given ſtraight line, which is not greater than the diameter.

Let the given circle be ABC; and D the given ſtraight line, not greater than the diameter of the circle: it is required to apply in the circle ABC a ſtraight line equal to the ſtraight line D.

Let BC the diameter of the circle ABC be drawn: if therefore BC be equal to D, the thing required has been done: for the ſtraight line BC has been applied, in the circle ABC, equal to the ſtraight line D. But if not, BC is greater than D (by ſupp.); and (by 3. 1.) make CE equal to D; and with the center C, and at the diſtance CE let the circle AEF be deſcribed; and let CA be joined.



Wherefore becauſe the point C is the center of the circle AEF CA is equal to CE; but D is equal to CE; therefore alſo D is equal to CA.

Wherefore in the given circle ABC, the ſtraight line AC has been applied, equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done.

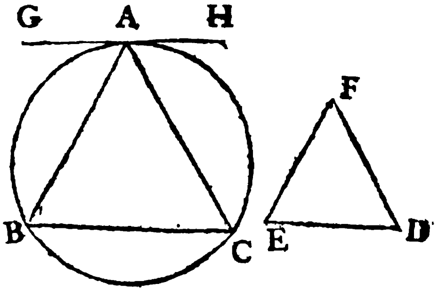
PROP. II.

In a given circle, to inſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe in the circle ABC a triangle equiangular to the triangle DEF.

Let GAH be drawn, touching the circle ABC in the *point* A; and let the angle HAC be made with the ſtraight line AH, and at the *point* A in it, equal to the angle DEF: and the *angle* GAB, with the ſtraight line AG, and at the *point* A in it, equal to the angle FDE; and let BC be joined.

Therefore becauſe a certain ſtraight line HAG touches the circle ABC, and a certain *ſtraight line* AC hath been drawn from the contact *cutting it*; therefore (by 32. 3.) the angle HAC is equal to the *angle* ABC, the angle in the alternate ſegment of the circle: but the angle HAC is equal to the angle DEF; therefore the angle ABC is equal to the *angle* FED. Certainly for the ſame reaſon alſo the angle ACB is equal to FDE; and therefore (by 32. 1.) the remaining angle BAC is equal to the remaining angle EFD: wherefore the triangle ABC is equiangular to the triangle DEF; and it has been inſcribed in the circle ABC (by def. 3. 4.).



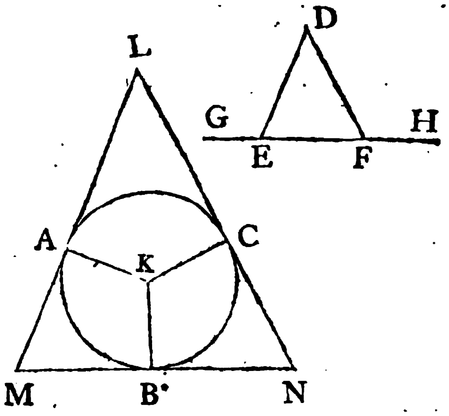
Wherefore a triangle equiangular to the given triangle hath been inſcribed in the given circle. Which was to be done.

PROP. III.

To circumſcribe a triangle, about a given circle, equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to circumſcribe a triangle about the circle ABC equiangular to the triangle DEF.

Let EF be produced towards both parts, to the points H, G; and let K the center of the circle ABC be taken; and let the ſtraight line KB be drawn as it may happen: and let the angle BKA be made with the ſtraight line KB and at the point K in it equal to the angle DEG; and BKC equal to the angle DFH; and through the points A, B, C let the *ſtraight lines* LAM, MBN, NCL be drawn touching the circle ABC.



And becauſe LM, MN, NL touch the circle ABC in the points, A, B, C; and from the center K; KA, KB, KC have been drawn to the points A, B, C; therefore (by 18. 3.) the angles at the points A, B, C are right angles. And becauſe the four angles of the quadrilateral figure AMBK are equal to four right angles (by 32. 1.); [ſince the quadrilateral figure AMBK is diviſible into two triangles] of which the angles KAM, KBM are two right angles; wherefore the remaining angles AKB, AMB are equal to two right angles; but the *angles* DEG, DEF are alſo (by 13. 1.) equal to two right angles; wherefore the *angles* AKB, AMB are equal to the *angles* DEG, DEF; of which AKB is (by conſt.) equal to DEG; therefore the remaining *angle* AMB is equal to the remaining *angle* DEF. Certainly in the ſame manner it will be demonſtrated that the *angle* LNM is equal to the *angle* DFE; therefore the remaining *angle* MLN is equal to the remaining *angle* EDF (by 32. 1.); therefore the triangle LMN is equiangular to the triangle DEF and (by def. 4. 4.) it is circumſcribed about the circle ABC.

Wherefore a triangle, has been circumſcribed about the given circle, equiangular to the given triangle. Which was to be done.

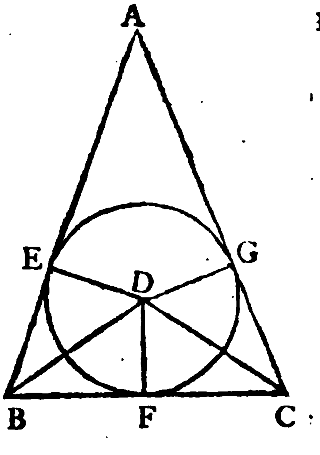
PROP. IV.

To inſcribe a circle in a given triangle.

Let ABC be the given triangle; it is required to inſcribe a circle in the triangle ABC.

Let the angles ABC, BCA be cut in halves by the ſtraight lines BD, CD; and let them meet one another in the point D; and from the *point* D let DE, DF, DG be drawn perpendiculars to the ſtraight lines AB, BC, CA.

And becauſe the angle ABD is equal to the angle CBD, for ABC is cut in halves; alſo the right angle BED is eqnal to the right



right angle BFD; therefore there are two triangles EBD, DBF having two angles equal to two angles, and one ſide equal to one ſide; viz. BD common to both extended under one of the equal angles; alſo (by 26. 1.) they will therefore have the remaining ſides equal to the remaining ſides; therefore DE *is* equal to DF: Certainly for the ſame reaſon alſo DG is equal to DF; wherefore the circle deſcribed with the center D and at the diſtance of *any* one of the *lines* DE, DF, DG will paſs alſo through the remaining points, and will touch the ſtraight lines AB, BC, CA; on account of the angles at the points E, F, G being right angles; for if it ſhall cut them; there will be a *ſtraight line,* drawn at right angles to the diameter from *its* extremity, falling within the circle, which (by 16. 3.) is abſurd: wherefore the circle deſcribed with the center D and at the diſtance of *any* one of the *lines* DE, DF, DG does not cut the ſtraight lines AB, BC, CA; therefore it will touch them; and will be a circle inſcribed in the triangle ABC.

Wherefore the circle EFG is inſcribed in the given triangle ABC. Which was to be done.

PROP. V.

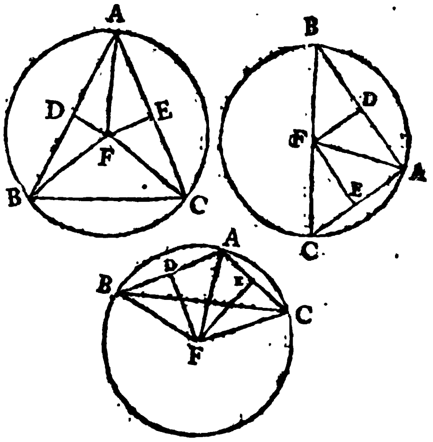
To circumſcribe a circle about a given triangle.

Let ABC be the given triangle: it is required to circumſcribe a circle about the given triangle ABC.

Let AB, AC be cut in halves in the points D and E; and let DF and EF be drawn from the points D, E at right angles to AB, AC; they will meet either within the triangle ABC or in the ſtraight line BC or without the triangle ABC.

Firſt let them meet within at the point F; and let BF, FC, FA be joined: and becauſe AD is equal to DB; and DF common and at right angles; therefore (by 4. 1.) the baſe AF is equal to the baſe FB: Certainly in like manner we ſhall demonſtrate that CF is alſo equal to FA; ſo that (by com. not. 1.) BF *is* alſo equal to

FC;



FC; therefore the three ſtraight lines FA, FB, FC are equal to one another; wherefore a circle deſcribed with the center F and at the diſtance of *any* one of the *lines* FA, FB, FC will alſo paſs through the remaining points; and the circle will be circumſcribed about the triangle ABC; and let it be deſcribed as the circle ABC.

But let DF, EF meet in the ſtraight line BC, as it is in the ſecond figure; and let AF be joined: Certainly in the ſame manner we ſhall demonſtrate that the point F is the center of the circle circumſcribed about the triangle ABC.

But let DF, EF meet without the triangle ABC, again in the point F; as it is in the third figure; and let AF, FB, FC be joined; and again becauſe AD is equal to DB, and DF common and at right angles; therefore (by 4. 1.) the baſe AF is equal to the baſe FB: Certainly in the ſame manner we ſhall demonſtrate that CF is equal to FA; ſo that alſo BF is equal to FC; therefore again the circle deſcribed with the center F and at the diſtance of *any* one of the *lines* FA, FB, FC will alſo paſs through the remaining points; and will be circumſcribed about the triangle ABC; and let it be deſcribed as ABC.

Wherefore a circle has been circumſcribed about the given triangle. Which was to be done.

Cor. And *it is* manifeſt that, when the center of the circle falls within the triangle, the angle BAC being in a ſegment greater than a ſemicircle, is leſs than a right angle; but when *it falls* in BC being in a ſemicircle, it will be a right angle; but when the center falls without the triangle ABC, the angle BAC being in a ſegment leſs than a ſemicircle, is greater than a right angle. So that alſo when the given triangle is acute angled the ſtraight lines DF, EF will meet within the triangle; but when BAC is a right angle; *they will meet* in BC; but when it is greater than a right angle, without the triangle ABC.

PROP.

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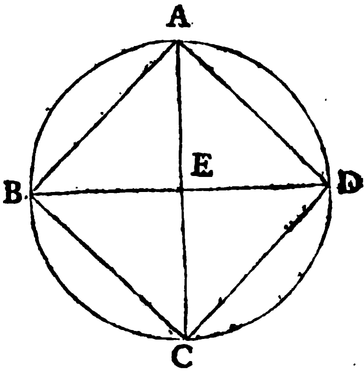
PROP. VI.

To inſcribe a ſquare in a given circle.

Let ABCD be the given circle: it is required to inſcribe a ſquare in the circle ABCD.

Let AC, BD, diameters of the circle ABCD, be drawn at right angles to one another; and let AB, BC, CD, DA be joined.



And becauſe BE is equal to ED; for E is the center; and EA common and at right angles; therefore (by 4. 1.) the baſe AB is equal to the baſe AD: certainly for the ſame reaſon alſo each of the *lines* BC, CD is equal to each of the *lines* BA, AD; therefore the quadrilateral figure ABCD is equilateral. I ſay it is alſo rectangular: for becauſe the ſtraight line BD is a diameter of the circle ABCD; therefore BAD is a ſemicircle; therefore the angle BAD is a right angle (by 31. 3.): Certainly for the ſame reaſon alſo each of the *angles* ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular: but it has been demonſtrated to be equilateral; therefore it is a ſquare; and it has been inſcribed in the given circle ABCD.

Wherefore the ſquare ABCD has been inſcribed in the given circle ABCD. Which was to be done.

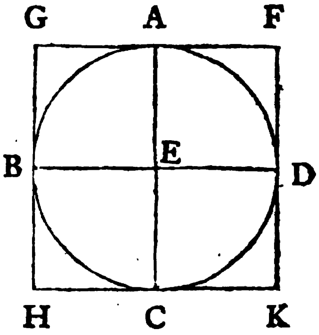
PROP. VII.

To circumſcribe a ſquare about a given circle.

Let ABCD be the given circle: it is required to circumſcribe a ſquare about the circle ABCD.

Let AC, BD two diameters of the circle ABCD be drawn at right angles to one another; and through the points A, B, C, D let FG, GH, HK, KF be drawn touching the circle ABCD.

Wherefore ſince FG touches the circle ABCD; and from the center E to the contact at A, the *ſtraight line* AE hath been drawn; therefore the angles at A are right angles: Certainly for the ſame reaſon



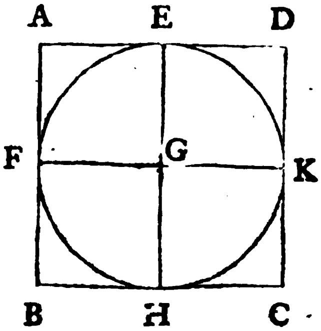
reaſon the angles at the points B, C, D are right angles; and becauſe the angle AEB is a right angle (by conſt.); as alſo the angle EBG (by cor.to 16. 3.) is a right angle; therefore (by 29.1.) GH is parallel to AC: Certainly for the ſame reaſon alſo AC is parallel to FK: and in the ſame manner we ſhall demonſtrate that each of the *lines* GF, HK is parallel to BED: wherefore GK, GC, AK, FB, BK are parallelograms; therefore (by 34. 1.) GF is equal to HK; and GH to FK; and becauſe AC is equal to BD; but AC (by 34. 1.) is equal to either of the *lines* GH, FK; and BD is equal to either of the *lines* GF, HK; therefore alſo each of the *lines* GH, FK is equal to each of the *lines* GF, HK; therefore the quadrilateral figure FGHK is equilateral: I ſay that it is alſo rectangular: For becauſe GBEA is a parallelogram; and AEB is a right angle; therefore (by 34. 1.) AGB *is* alſo a right angle; certainly in the ſame manner we ſhall demonſtrate, that the angles at the *points* H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular; and it has been alſo demonſtrated to be equilateral; therefore it is a ſquare and has been circumſcribed about the circle ABCD.

Wherefore a ſquare has been circumſcribed about the given circle. Which was to be done.

PROP. VIII.

To inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare: it is required to inſcribe a circle in the ſquare ABCD.



Let each of the *lines* AB, AD be cut in halves at the points F, E; and through the point E, let EH be drawn parallel to either of the *lines* AB, CD; and through the point F, let FK be drawn parallel to either of the *lines* AD, BC; therefore each of the *figures* AK, KB; AH, HD; AG, GC; BG, GD is a parallelogram; and certainly (by 34. 1.) the oppoſite ſides of them are equal: and becauſe AD is equal to AB (by ſupp.); and AE is the half of AD; and AF the half of AB; therefore AE is equal to AF; ſo that alſo the oppoſite *ſides* are equal; therefore FG is equal to GE: Certainly in the ſame manner we ſhall demonſtrate that each of the *lines* GH, GK is equal to each of the lines FG, GE; therefore the four GE, GF, GH, GK are equal to one another; therefore the circle deſcribed with G for a center and at the diſtance of *any* one of the *lines* GE, GF, GH, GK will alſo paſs through the remaining points; and will touch the ſtraight lines AB, BC, CD, DA; on account of the angles at the *points* E, F, H, K being right angles (by cor. to 16. 3.); for if the circle will cut the *ſtraight lines* AB, BC, CD, DA; the ſtraight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle; which is abſurd (by 16. 3.); wherefore the circle deſcribed with the center G and at the diſtance of *any* one of the *lines* GE, GF, GH, GK does not cut the ſtraight lines AB, BC, CD, DA; therefore it will touch them and will be inſcribed in the ſquare ABCD.

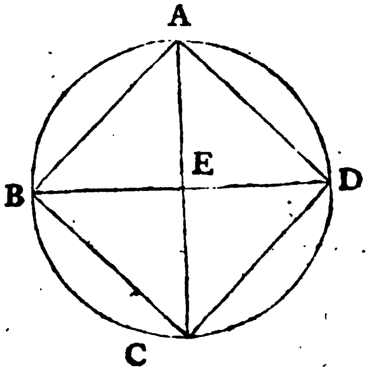
Wherefore a circle has been inſcribed in a given ſquare. Which was to be done.

PROP. IX.

To circumſcribe a circle about a given ſquare.

Let ABCD be the given ſquare: it is required to circumſcribe a circle about the ſquare ABCD.

For AC and BD being joined let them cut one another in the point E.



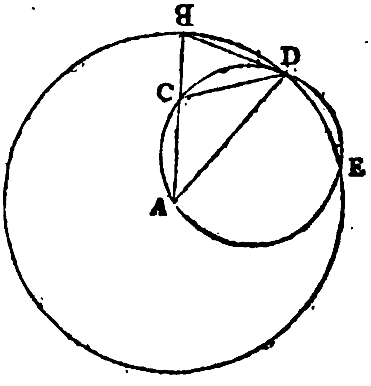
And becauſe DA is equal to AB; and AC common; certainly the two DA, AC are equal to the two BA, AC and the baſe DC is equal to the baſe BC; therefore (by 8. 1.) the angle DAC is equal to the angle BAC; therefore the angle DAB has been cut in halves by AC: Certainly in the ſame manner we ſhall demonſtrate that each of the angles ABC, BCD, CDA hath been cut in halves by the ſtraight lines AC, DB: and becauſe the angle DAB is equal to ABC; and EAB is the half of DAB; and EBA *is* the half of ABC; therefore the *angle* EAB is equal to the *angle* EBA; ſo that alſo (by 6. 1.) the ſide EA is equal to the ſide EB: Certainly in the ſame manner we ſhall demonſtrate that each of the ſtraight lines EC, ED is equal to each of the ſtraight lines EA, EB; therefore the four *ſtraight lines* EA, EB, EC, ED are equal to one another; wherefore a circle deſcribed with the center E and at the diſtance of *any* one of the lines EA, EB, EC, ED will paſs alſo through the remaining points; and will be circumſcribed about the ſquare ABCD.

Wherefore a circle has been circumſcribed about a given ſquare. Which was to be done.

PROP. X.

To make an iſoſceles triangle, having each of the angles at the baſe double of the remaining *angle*.

Draw any ſtraight line AB; and let it be cut in the point C, (by 11. 2.) ſo that the rectangle contained by AB, BC may be equal to the ſquare of CA; and with the center A and at the diſtance AB let the circle BDE be deſcribed: and let BD a ſtraight line equal to the ſtraight line AC, which is not greater than the diameter of the circle BDE be applied in the circle BDE (by 1. 4.): and let DA, DC be joined; and let the circle ACD be circumſcribed (by 5. 4.) about the triangle ACD.



And becauſe the *rectangle contained* by AB, BC is equal to the *ſquare* of AC: and AC is equal to BD; wherefore the *rectangle contained* by AB, BC is equal to the ſquare of BD; and becauſe a certain point B has been taken without the circle ACD; and from the *point* B, two ſtraight lines BCA, BD have fallen upon the circle ACD and one of them cuts *it*, and the other falls upon *it*; and the *rectangle contained* by AB, BC is equal to the *ſquare* of BD; wherefore (by 37. 3.) BD touches the circle ACD; wherefore becauſe BD touches *it*, and from the contact at D: DC has been drawn *cutting it*; therefore (by 32. 3.) the angle BDC is equal to the angle DAC in the alternate ſegment of the circle: wherefore becauſe the *angle* BDC is equal to DAC let the *angle* CDA which is common be added; therefore the whole *angle* BDA is equal to the two *angles* CDA, DAC: But (by 32. 1.) the outward *angle* BCD is equal to the *angles* CDA, DAC; therefore the *angle* BDA is equal to the *angle* BCD: But the *angle* BDA is equal to CBD (by 5. 1.); ſince the ſide AB is equal to AD; ſo that alſo the *angle* DBA is equal to the *angle* BCD: therefore the three *angles* BDA, DBA, BCD are equal to one another: and ſince the angle DBC is equal to BCD (by 6. 1.) the ſide BD is alſo equal to the ſide DC; but BD is put equal to CA; therefore alſo AC is equal to CD; ſo that the angle CDA is equal to DAC; therefore the angles CDA, DAC are the double of DAC; but the *angle* BCD is equal to the *angles* CDA, DAC; therefore BCD is alſo the double of DAC; but BCD is equal to either of the *angles* BDA, DBA; therefore each of the *angles* BDA, DBA is double of the *angle* DAB.

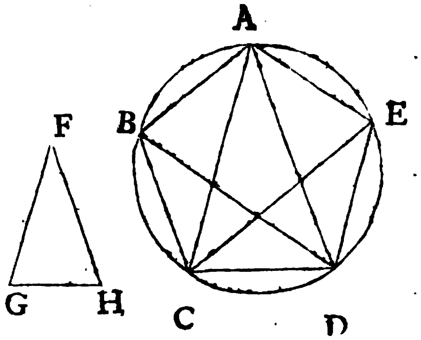
Therefore an iſoſceles triangle ADB has been made, having each of the angles at the baſe BD double of the remaining *angle*. Which was to be done.

PROP. XI.

To inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

Make (by 10. 4.) the iſoſceles triangle FGH having each of the angles at G and H double of the angle at F: and inſcribe (by 2. 4.) the triangle ACD in the circle ABCDE equiangular to the triangle FGH; ſo that the *angle* CAD may be equal to the angle at F; and each of the *angles* at G and H equal to each of the angles ACD, CDA; therefore alſo each of the *angles* ACD, CDA is double of the *angle* CAD: Let each of the *angles* ACD, CDA be cut in halves (by 9. 1.) by the ſtraight lines CE, DB; and let AB, BC, DE, EA be drawn.



Wherefore becauſe each of the angles ACD, CDA is double of the angle CAD; and they have been cut in halves by the ſtraight lines CE, DB; therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but (by 26. 3.) equal angles ſtand upon equal circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another: but (by 29. 3.) equal ſtraight lines are extended under equal circumferences; therefore the five ſtraight lines AB, BC, CD, DE, EA are equal to one another; therefore the pentagon ABCDE is equilateral; I ſay it is alſo equiangular; for becauſe the circumference AB is equal to the circumference DE; let the common *circumference* BCD be added; therefore the whole circumference ABCD is equal to the whole circumference EDCB: And the angle AED ſtands upon the circumference ABCD; and the angle BAE *ſtands* upon the circumference EDCB; and therefore (by 27. 3.) the angle BAE is equal to the angle AED: Certainly for the ſame reaſon alſo each of the angles ABC, BCD, CDE are equal to either of the angles BAE, AED: therefore the pentagon ABCDE is equiangular; but it has alſo been demonſtrated to be equilateral.

Wherefore an equilateral and equiangular pentagon has been inſcribed in a given circle. Which was to be done.

PROP. XII.

To circumſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to circumſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let A, B, C, D, E be underſtood to be the points of the angles of a pentagon inſcribed (by 11. 4.) ſo that the circumferences AB, BC, CD, DE, EA may be equal: And through the *points* A, B, C, D, E let GH, HK, KL, LM, MG be drawn touching the circle: and let F the center of the circle ABCDE be taken; and let FB, FK, FC, FL, FD be joined.

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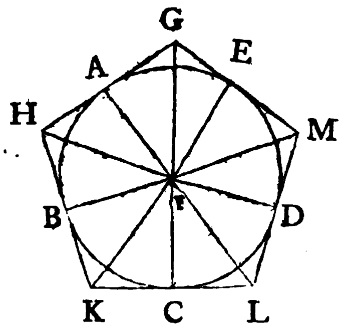


And becauſe the ſtraight line KL touches the circle ABCDE in the *point* C; and from the center F to the contact at C the *line* FC hath been drawn; therefore (by 18. 3.) FC is perpendicular to KL; therefore each of the angles at the point C is a right angle. For the ſame reaſon alſo the angles at the points B, D are right angles: and becauſe the angle FCK is a right angle; therefore (by 47. 1.) the *ſquare* of FK is equal to the *ſquares* of FC, CK: Certainly for the ſame reaſon alſo the *ſquare* of FK is equal to the *ſquares* of FB, BK; therefore (by com. not. 1.) the *ſquares* of FC, CK are equal to the *ſquares* of FB, BK; of which the *ſquare* of FC is equal to the *ſquare* of FB; therefore the remaining *ſquare* of CK is equal to the remaining *ſquare* of BK; therefore BK is equal to CK: And becauſe FB is equal to FC and FK common; certainly the two BF, FK are equal to the two CF, FK; and the baſe BK is equal to the baſe CK; therefore (by 8. 1.) the angle BFK is equal to the angle CFK; and the *angle* BKF; equal to FKC; therefore the *angle* BFC is double of KFC and BKC *is double* of FKC. Certainly for the ſame reaſon alſo CFD is double of CFL; and the *angle* CLD *is double* of CLF. And becauſe the circumference BC is equal to the *circumference* CD (by the conſt.); the angle BFC is alſo equal (by 27. 3.) to the *angle* CFD; and BFC is the double of KFC; and the *angle* DFC is the double of LFC; therefore the *angle* KFC is equal to CFL; certainly FKC, FLC are two triangles having the two angles equal to the two angles, each to each; and one ſide equal to one ſide; FC common to them *both*; therefore (by 26. 1.) they will alſo have the remaining ſides equal to the remaining ſides; and the remaining angle *equal* to the remaining angle; therefore the ſtraight line KC is equal to CL; and the angle FKC to FLC: and ſince KC is equal to CL; therefore KL is the double of KC. Certainly in the ſame manner HK will be demonſtrated to be double of BK: and becauſe BK has been demonſtrated to be equal to KC; and KL is the double of KC; and HK the double of BK; therefore HK is equal to KL; certainly in the ſame manner each of the *lines*

GH, GM, ML will be demonſtrated to be equal to either of the *lines* HK, KL; therefore the pentagon GHKLM is equilateral. I ſay that it is alſo equiangular; for becauſe the angle FKC is equal to FLC; and HKL has been demonſtrated to be double of FKC; and KLM to be double of FLC; therefore HKL is alſo equal to KLM: certainly in the ſame manner each of the *angles* KHG, HGM, GML will be demonſtrated to be equal to either of the *angles* HKL, KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH are equal to one another: therefore the pentagon GHKLM is equiangular; but it has been alſo demonſtrated to be equilateral; and it has been circumſcribed about the given circle ABCDE. Which was to be done.

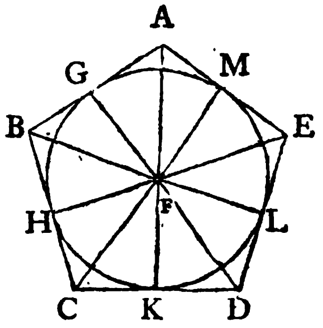
PROP. XIII.

To inſcribe a circle, in a given pentagon which is equilateral and equiangular.

Let ABCDE be the given pentagon, which is equilateral and equiangular: it is required to inſcribe a circle in the pentagon ABCDE.

Let each of the angles BCD, CDE be cut in halves, by each of the ſtraight lines CF, DF; and from the point F, in which the ſtraight lines CF, DF meet one another, let the ſtraight lines FB, FA, FE be drawn. And becauſe BC, CD are equal (by ſupp.); and CF common; certainly the two BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF (by conſt.); therefore (by 4. 1.) the baſe BF is equal to the baſe DF; and the triangle BFC is equal to the triangle DCF; and the remaining angles are equal to the remaining angles, under which the equal ſides are extended; therefore the angle CBF is equal to the angle CDF: and becauſe the *angle* CDE is double of CDF; but CDE is equal (by ſupp.) to ABC; and the *angle* CDF to CBF; therefore CBA is the double of CBF; therefore the angle ABF is equal to FBC: therefore the angle ABC is cut in halves by the ſtraight line BF. Certainly in the ſame manner it will be demonſtrated, that each of the angles BAE, AED hath been cut in halves by each of the ſtraight lines FA, FE.

Let



Let the perpendiculars FG, FH, FK, FL, FM be drawn from the point F to the ſtraight lines AB, BC, CD, DE, EA; and becauſe the angle HCF is equal to KCF; and the right angle FHC is equal to the right angle FKC; certainly there are two triangles FHC, FKC having two angles equal to two angles; and one ſide equal to one ſide, viz. FC common to them *both*, extended under one of the equal angles; therefore (by 26. 1.) they will have the remaining ſides equal to the remaining ſides; therefore the perpendicular FH *is* equal to the perpendicular FK: certainly in the ſame manner it will be demonſtrated, that each of the *lines* FL, FM, FG is equal to either of the *lines* FH, FK; therefore the five ſtraight lines FG, FH, FK, FL, FM are equal to one another: wherefore the circle deſcribed with the center F and at the diſtance of *any* one of the *lines* FG, FH, FK, FL, FM will alſo paſs through the remaining points and will touch the ſtraight lines AB, BC, CD, DE, EA; on account of the angles at the points G, H, K, L, M being right angles; for if it will not touch them, but will cut them; it will happen that a *ſtraight line* drawn at right angles to a diameter from its extremity falls within the circle, which (by 16. 3.) has been demonſtrated to be abſurd: therefore the circle deſcribed with the center F and at the diſtance of *any* one of the *lines* FG, FH, FK, FL, FM will not cut the ſtraight lines AB, BC, CD, DE, EA; therefore it will touch them: let it be deſcribed as GHKLM.

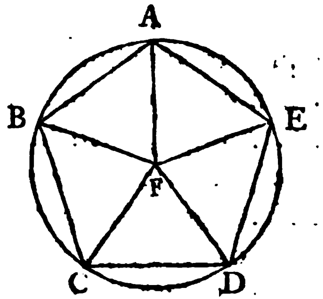
Wherefore a circle has been inſcribed in a given pentagon, which is equilateral and equiangular. Which was to be done.

PROP. XIV.

To circumſcribe a circle about a given pentagon, which is equilateral and equiangular.

Let ABCDE be the given pentagon, which is equilateral and equiangular; it is required to circumſcribe a circle about the pentagon ABCDE.

Let each of the angles BCD, CDE be cut in halves by each of the ſtraight lines CF, DF; and from the point F in which the ſtraight lines meet, let the ſtraight lines FB, FA, FE be drawn to the points B, A, E; certainly in the ſame manner as in the *propoſition* before this it will be demonſtrated, that each of the angles CBA, BAE, AED hath been cut in halves by each of the ſtraight lines BF, FA, FE.



And becauſe the angle BCD is equal to the angle CDE; and the *angle* FCD is the half of BCD; and the *angle* CDF is the half of CDE; therefore alſo the *angle* FCD is equal to FDC: ſo that (by 6. 1.) the ſide FC is equal to the ſide FD. Certainly in the ſame manner it will be demonſtrated that each of the *lines* FB, FA FE is equal to either of the *lines* FC, FD: therefore the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; wherefore the circle deſcribed with the center F and at ahe diſtance of *any* one of the lines FA, FB, FC, FD, FE will alſo paſs through the remaining points; and will be circumſcribed about the pentagon ABCDE, which is equilateral and equiangular; let it be circumſcribed and let it be ABCDE.

Wherefore a circle has been circumſcribed about a given pentagon, which is equilateral and equiangular. Which was to be done.

PROP. XV.

To inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in the circle ABCDEF.

Let AD the diameter of the circle ABCDEF be drawn: and let G the center of the circle be taken; and with the center D and at the diſtance DG let the circle EGCH be deſcribed; and EG, CG being joined, let them be produced to the points B, F; and let AB, BC, CD, DE, EF, FA be drawn; I ſay that the hexagon ABCDEF is equilateral and equiangular.

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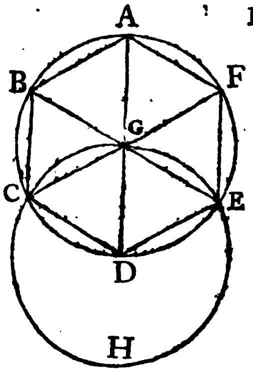


For becauſe the point G is the center of the circle ABCDEF, the *ſtraight line* GE is equal to GD: again becauſe the point D is the center of the circle EGCH, the *line* DE is equal to GD; but GE has been demonſtrated to be equal to GD; therefore GE is equal to ED; therefore the triangle EGD is equilateral; and therefore the three angles of it EGD, GDE, DEG are equal to one another, ſince the angles at the baſe of iſoſceles triangles are equal to one another; and (by 32. 1.) the three angles of a triangle are equal to two right angles; therefore the angle EGD is the third *part* of two right angles: Certainly in the ſame manner it will be demonſtrated that the angle DGC is the third *part* of two right angles; and becauſe the ſtraight line CG ſtanding upon EB makes the adjacent angles EGC, CGB equal to two right angles; therefore the remaining *angle* CGB is the third *part* of two right angles; wherefore the angles EGD, DGC, CGB are equal to one another; ſo that (by 15. 1.) the vertical angles to them; viz. BGA, AGF, FGE are equal to EGD, DGC, CGB; therefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another; but (by 26. 3.) equal angles ſtand upon equal circumferences; therefore the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another; but (by 29. 3.) equal ſtraight lines are extended under equal circumferences; therefore the ſix ſtraight lines are equal to one another; therefore the hexagon ABCDEF is equilateral: I ſay it is alſo equiangular; for becauſe the circumference AF is equal to the circumference ED; let the common circumference ABCD be added; therefore the whole circumference FABCD is equal to the whole circumference EDCBA; and the angle FED ſtands upon the circumference FABCD; and the angle AFE upon EDCBA; therefore (by 27. 3) the angle AFE is equal to the *angle* DEF; in like manner it will be demonſtrated that the remaining angles of the hexagon ABCDEF, one by one, are equal to either of the angles AFE, FED; therefore the hexagon ABCDEF is equiangular; and it has been demonſtrated to be equilateral; and it has been inſcribed in the circle ABCDEF.

Wherefore an equilateral and equiangular hexagon has been inſcribed in a given circle. Which was to be done.

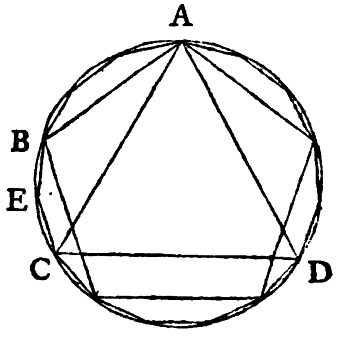
Cor. From this *it is* manifeſt, that the ſide of a hexagon is equal to the *line* from the center of the circle.

And if we draw through the points A, B, C, D, E, F *ſtraight lines* touching the circle, an equilateral and equiangular hexagon will be circumſcribed about the circle, agreeable to what has been ſaid concerning the pentagon: And farther we ſhall inſcribe in a given hexagon and alſo circumſcribe a circle by the like *ſteps* as have been mentioned concerning the pentagon.

PROP. XVI.

To inſcribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.



Let AC, be inſcribed in the circle ABCD, the ſide of an equilateral triangle inſcribed in it; and let AB, be *inſcribed,* the ſide of an equilateral pentagon. Wherefore of what equal ſegments the circle ABCD is fifteen; of ſuch the circumference ABC, being the third *part* of the circle, will be five; and the circumference AB, being the fifth *part* of the circle, will be three; therefore the remainder BC will be two of theſe equal *ſegments*; let BC be cut in halves in the point E (by 30. 3.); therefore either of the circumferences BE, EC is the fifteenth *part* of the circle ABCD; therefore joining the ſtraight lines BE, EC; if we apply in the circle ABCD continually ſtraight lines equal to them (by 1. 4.) there will be inſcribed in it an equilateral and equiangular quindecagon. Which was to be done.

And in like manner as was done concerning the pentagon; if through the diviſions of the circle we draw *lines* touching the circle; an equilateral and equiangular quindecagon will be circumſcribed about the circle; and farther we ſhall inſcribe in a given quindecagon, which is equilateral and equiangular, and alſo circumſcribe a circle by the like *ſteps* as have been mentioned concerning the pentagon.

DISSERTATION VI.

IN the former diſſertation many things are not mentioned which a careleſs reader requires to be put in mind of, becauſe I conſidered that ſuch readers would ſtand in need of freſh admonitions, to induce them to read my obſervations; which determined me to ſteer a kind of middle courſe, by reducing my remarks to ſome general heads, and illuſtrating them by examples, which the reader is ſuppoſed to apply in all ſimilar circumſtances. Upon this principle it ſeemed needleſs to remark that the demonſtrations of the twelfth, thirteenth and fourteenth propoſitions of the fourth book, are more general than they appear to be at firſt ſight; for if the reader obſerve my general rule for the examination of every ſuppoſition; he will find no conſequence drawn from the number of ſides being five; it being only neceſſary that the inſcribed figures be equilateral and equiangular; which ſhews the demonſtrations to be much more general than is profeſſed in the propoſitions. In like manner, as it has been particularly remarked before, it ſeemed unneceſſary to deſire the ſtudent to obſerve the ingenious contrivance for cutting the angles of the figure in halves, in the thirteenth propoſition, having earneſtly recommended an attention to all ſuch indirect conſtructions: for in this inſtance if all the angles were cut in halves directly, it would be found no eaſy matter to prove that the lines will all meet in F; but by cutting any two of the angles in halves, which follow each other in order, the point F is fixed; and then joining this point, and the other angular points, it is eaſy to prove that theſe lines will cut all the other angles in halves. And whoever reads the former diſſertations with this allowance, will readily grant that I have been ſufficiently particular. I now proceed to a ſubject which will exerciſe the reader's patience if he chuſes to go along with me; for this diſſertation will conſiſt of a minute enquiry into the origin of our ideas of proportional magnitudes.

CHAP. I.

Of parts and multiples.

IN the firſt four books Euclid conſiders no other relation of magnitudes but their equality; at leaſt when he ſpeaks of anyother, it is in a looſe and undetermined manner, without ever conſidering how much greater or how much leſs the one is than the other. And this will be obvious to any one who chuſes to turn his thoughts to the nature of his propoſitions; the common notion of whole and part therefore is ſufficiently accurate for his purpoſe in theſe books. But the object which he has in view in this book, being to ſettle other relations of magnitudes beſides their equality, makes it neceſſary to introduce, and conſequently to define, a new idea of whole and part under the name of a multiple and part. Take two ſtraight lines, one of them fixt and the other undetermined; by the third propoſition of the firſt book two unequal lines being given, you may cut off a part from the greater equal to the leſs; and thus you may make a ſtraight line, twice, three times, fourtimes, or fivetimes &c. as long as another; the line originally fixt is called a part; and the line which you determine by this conſtruction is called a multiple; they are relative terms, or as Euclid expreſſes it, a magnitude of a magnitude. The firſt thing therefore which the ſtudent has to ſettle in his own mind, is, what the magnitudes are and in what circumſtances they muſt be, before this relation of multiple and part can take place; and not only ſo, but what the magnitudes are between which he can exhibit theſe relations ſcientifically from what has been demonſtrated in the firſt four books. And firſt it is obvious that a ſtraight line cannot be a part or multiple of a triangle; nor of the circumference of a circle; nor indeed can we conſider one triangle as a part of another unleſs they be between the ſame parallel lines; nor the circumferences of two circles; as part and multiple unleſs the circles be equal: but to ſettle this more particularly let us conſider what magnitudes, we can multiply, that is meaſure exactly, and in what circumſtances: Now we can multiply or meaſure exactly, two ſtraight lines by the third propoſition of the firſt book; that is we can take any ſtraight line five, ſix &c. times; or if the ſuppoſition be that one ſtraight line is five, ſix &c. times any other line; we can divide this multiple into its parts. But the reader is not to take my word for this; and the more inſtances in which he tries it, the more likely is he to underſtand what follows.

By multiplying (that is doubling, trebling &c.) the baſe of a triangle, you multiply the triangle, provided the part and multiple are between the ſame parallels; and the ſame may be ſaid of parallelograms: ſo that here we begin to make our notions of magnitudes ſomewhat more complicated. It is ſaid in the firſt book that triangles upon equal baſes and between the ſame parallels are equal, but now we advance a ſtep farther, and ſay that triangles upon double baſes are double, and upon treble baſes are treble &c; which tranſition the ſtudent who is deſirous to have his ideas keep pace with his apparent progreſs in the ſcience will do well to obſerve.

Angles, circumferences and ſectors of equal circles may be multiplied by the principles contained in the twenty-ſixth-ſeventh-eighth, and-ninth propoſitions of the third book: becauſe by placing or applying equal ſtraight lines in a circle you cut off equal circumferences; you make equal angles at the center or circumference; and it is alſo eaſy to prove that the ſectors are equal: all this however the ſtudent muſt demonſtrate; and I believe it will be found that theſe are all the magnitudes which can be multiplied according to Euclid's idea in his two firſt definitions; for it does not appear to me that it would be ſufficient to have one triangle three times as large as another unleſs they be between the ſame parallel lines, to ſay that the one is a multiple of the other; becauſe the definitions ſuppoſe that the multiple may be divided into its parts, which is what I underſtand by the word *meaſures*. But, all theſe, the ſtudent ſhould examine by a particular conſtruction, if he means to underſtand any part of this buſineſs; and I am perſuaded he will find, that ſtraight lines; triangles and parallelograms between the ſame parallel lines; angles, circumferences and ſectors of equal circles are all the magnitudes which can have the relation of multiple and part to one, who has no other principles to go upon but thoſe contained in the firſt four books.

It may be thought that particular inſtances explaining each of theſe circumſtances would have been uſeful; but the conſtruction of ſuch figures is extremely obvious, and it will be much more improving for the learner to deſcribe the figures for himſelf, becauſe he may look at figures already deſcribed, without comprehending what they are intended to illuſtrate, which can never be the caſe if the figures are properly deſcribed by himſelf. Now this being perfectly underſtood, and the proper meaning of the two firſt definitions determined; the ſtudent is next to endeavour to acquire a ready uſe of them; by reaſoning upon ſuch conſequences as follow moſt directly from them: let him ſuppoſe one magnitude to be a part of another and try what conſequences will follow; and the only direct conſequence is that the part will meaſure the multiple; but what is the meaning of this? Let the reader ſuppoſe two ſtraight lines which we ſhall call AB and C; and let C be a part of AB; then if we begin at the point A and (by 3. 1.) take of a part equal to C and from the remainder a part equal to C; we muſt at laſt fall in with the point B; becauſe, if this did not happen, C could not be a part of AB, contrary to the ſuppoſition; becauſe this is what I underſtand by meaſuring it: and the ſame conſequence follows if we ſuppoſe AB a multiple of C.

But it will be neceſſary to reaſon in the ſame manner upon each of the magnitudes, which have been mentioned as coming under theſe definitions of a part and multiple.

CHAP. II.

Of equimultiples.

THE word *equimultiple*, one might imagine has its meaning fully aſcertained, by the common acception of the two words of which it is made up; and yet I cannot tell how it happens, that

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it hardly ever conveys a diſtinct idea to the mind of a learner, as it is applied in this ſcience; ſurely nothing ſeems more obvious than that it is to be underſtood of magnitudes taken an equal number of times; but I have generally found that there is ſome confuſed notion of equality of magnitude that accompanies the conception of equimultiples; than which nothing can poſſibly be more abſurd; for it means only that the magnitudes are taken the ſame number of times; thus if any ſtraight line, which you may call A be taken three times; and call this BC: and ſuppoſe any angle to be trebled; we ſay that BC and this angle, are equimultiples of the line A and of the angle which was taken three times; or to be more particular (turn to the figure for the firſt propoſition of the ſixth book) the ſtraight line HC and the triangle AHC, though they can have no connexion with each other as to magnitude, are nevertheleſs equimultiples of the ſtraight line BC and of the triangle ABC: and again (ſee the figure to the thirty third propoſition of the ſixth book) the circumference EN and the angle EHN, though magnitudes of quite different kinds, are equimultiples of the circumference EF and of the angle EHF; or in ſhort this expreſſion means no more, but that when the multiples are divided into their parts, the number of diviſions in each is the ſame; as in the inſtances above, HC is divided into three parts each equal to BC; and the triangle AHC into three parts alſo each equal to the triangle ABC; theſe equimultiples therefore agree in nothing but that they can be divided into the ſame number of parts; and this is the moſt direct conſequence which will follow from the ſuppoſition of their being equimultiples. But in order to underſtand the meaning of this term, and the two firſt definitions it will be proper for the reader to examine the three firſt propoſitions together with the fifth and ſixth of this book.

CHAP. III.

Containing further illuſtrations of the firſt and ſecond definitions.

THE reader will perceive, if he has gone through the different ſteps of the conſtructions which I have recommended, that all thoſe magnitudes which we can multiply; are doubled, trebled &c. by the aſſiſtance of a ſtraight line; ſo that whatever can be ſhewn to be a property of, a multiple, equimultiples or parts of a ſtraight line the ſame may be readily transferred to triangles or parallelograms, between the ſame parallel lines; to angles; circumferences and ſectors of equal circles; which I contend are all the magnitudes which we can multiply, agreeable to Euclid's two firſt definitions; and probably for this reaſon the conſtructions and demonſtrations in this fifth book are confined to ſtraight lines only; although the enunciations are expreſſed of magnitudes in general. Let us ſuppoſe that the reader has examined the three firſt propoſitions and the fifth and ſixth; only obſerving, that Euclid is here demonſtrating properties of ſtraight lines, I ſhall now proceed to make ſome remarks upon them in order. And firſt it is ſaid that AB is a multiple of E; *equimultiple you mean*; no, not ſo faſt let us examine one thing at a time; I ſay that the ſuppoſition is; AB is a multiple of E; well let us ſee what conſequence follows from this; cut off from AB the line AG equal to E; and by proceeding thus you muſt fall in with the point B; that is I thus divide AB into parts equal to E; or I meaſure AB; take care to diſtinguiſh between this, *and the ſuppoſing it divided*; for if the reader be given to ſuppoſitions of this kind, he had better ſuppoſe the whole book demonſtrated. In like manner CD by the ſuppoſition is a multiple of F; and the ſame conſequences follow; namely that in the ſame manner I divide CD into parts equal to F: But farther by the ſuppoſition AB and CD are equimultiples of E and F; let us ſee what conſequence follows from this part of the ſuppoſition; nothing more than this, that after I have divided the multiples into magnitudes equal to the parts, the number of diviſions in both is the ſame; for certainly if AB be the double of E and CD the double of F; AB contains E twice and CD contains F twice; and the ſame may be ſaid of any other equimultiples; and the uſe made of it in this propoſition is, that if you take away the parts one by one, when you have done with one multiple all the other equimultiples muſt be exhauſted at the ſame time.

But farther, this propoſition ſhould be examined ſtill more particularly; because, although this may ſeem needleſs for thoſe to whom Euclid writes, I know it to be eſſentially neceſſary for thoſe to whom this diſſertation is addreſſed. The ſtudent is therefore to bring this concluſion more directly to the definitions, at leaſt to take ſuch ſteps as ſeem moſt likely to carry his attention up to the definitions. For this purpoſe let him draw two indefinite ſtraight lines; and cutting off from one of them a line equal to E and F together; he is then to proceed in the ſame manner and cut off from the other a line equal to AG and CH together; and from the remainder a line equal to GB and HD together, and ſo on until he has made a line equal to AB and CD together: He is next to demonſtrate that the line, made equal to E and F together, will meaſure the line made equal to AB and CD together; and when this is done he can ſay, that the one line is a part or multiple of the other: But after all it remains to be proved that the one line is contained in the other, juſt as often as E is contained in AB; and then he may ſay that AB; and AB, CD together are equimultiples of E; and of E, F together. Let him, after this point is ſettled to his full conviction, proceed in the ſame manner with three multiples &c. until he can ſay upon good grounds if there be any number of magnitudes &c.

The ſecond and third propoſitions muſt be perfectly intelligible to any one who has gone the progreſs which I have recommended upon the firſt. But before entering upon the fifth it will be proper for the reader to obſerve that we are not as yet taught how to take even a third part of a ſtraight line, much leſs how to take any part: indeed we can cut a ſtraight line in halves; by which conſtruction we may take a half, a fourth part, and an eighth part &c. but nothing more: the conſtruction therefore in the fifth propoſition is impracticable; for if AB be three times CD, I cannot take a third part of BE but upon the ſuppoſition that EB is a multiple of FD; which is taking for granted the very thing to be proved: if a line be a multiple of another I can divide it into its parts; but not otherwiſe; which makes a different conſtruction neceſſary here, from that given in the text; but ſufficiently obvious from the conſtruction in the next propoſition: for producing BA directly forward, we may cut off lines equal to FD, until we have the ſame multiple of FD, that AB is of CD; and call this AG; then.

then by the firſt propoſition EG is the ſame multiple of CD that AB is of CD, therefore (by com. not 6.) AB and EG are equal; take away AE which is common, and EB is equal to AG; therefore EB is the ſame multiple of FD that AB is of CD.

It is not eaſy to conjecture how this blunder could have crept in, unleſs by the deception of a particular figure: for if AB be the double of CD, the conſtruction is practicable, becauſe then I divide EB in halves; and take CG equal to the half of it; and then indeed AB is the ſame multiple of GF as of CD, therefore GF and CD are equal &c. And it is obvious that if AB be taken four times CD the ſame conſtruction will hold; but it fails, if AB be taken three times, for then I have no principle upon which I could pretend to divide EB; unleſs I ſhould meaſure it by FD, which is taking the very thing for granted which we want to demonſtrate, namely that EB is a multiple of FD; and not only a multiple of it, *but the ſame multiple of it*, that AB is of CD; that is, I ſet out with a ſuppoſition that CG is equal to FD in order to prove them to be equal; which is certainly very bad reaſoning if it deſerve the name. I have been the more particular in explaining this, becauſe it ſeems well calculated to illuſtrate this doctrine of parts and multiples.

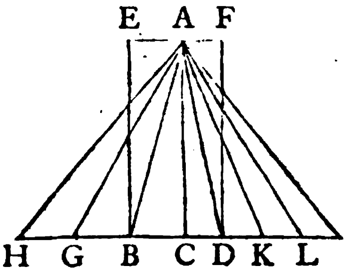
CHAP. IV.

Containing a compariſon of the multiples of four magnitudes.

LET ABC, ACD be two triangles between the ſame parallel lines; let equimultiples of the baſe BC and of the triangle ABC be taken; and alſo any equimultiples whatever of the baſe CD and of the triangle ACD; I ſay that if the multiple of BC exceed the multiple of CD; the multiple of the triangle ABC will exceed the multiple of the triangle ACD; if equal, equal; and if leſs, leſs.

Here the reader is to fix his attention upon four magnitudes, which he may diſtinguiſh thus, by calling BC the firſt, CD the ſecond, the triangle ABC the third, and the triangle ACD the fourth; according to our ſuppoſition then, we are to take equimultiples

multiples of the firſt and third at a venture; and alſo any equimultiples whatever of the ſecond and fourth. But obſerve we do not compare the *equimultiples* with one another; only the multiple of the firſt, with the multiple of the ſecond, which are both ſtraight lines; and the multiples of the third and fourth, which are both triangles. And firſt let us exhibit any multiple of BC the firſt at a venture; produce CB indefinitely and cut off as many lines as you pleaſe each equal to CB; as BG, GH &c; and let CH be a multiple of BC taken at a venture, which it is if we draw no conſequence from CH's containing CB any particular number of times. Now we are to take the triangle ABC the ſame number of times; and this is performed by joining AH, for the triangle AHC contains the triangle ABC juſt as often as CH contains CB; which may be proved by joining AG; for as many lines CB, BG, GH as there are, juſt ſo many triangles are there; and becauſe theſe ſtraight lines are equal; (by 38. 1.) the triangles are equal of which theſe lines are the baſes; and the number of triangles, and the number of ſtraight lines being the ſame; CH contains CB juſt as often as the triangle AHC contains the triangle ABC; therefore CH and the triangle AHC are equimultiples of CB and of the triangle ABC. Again we are to take equimultiples any whatever of the ſecond and fourth, of CD and of the triangle ACD: produce CD directly forward, and take DK, KL each equal to CD; and in the ſame manner CL is any multiple of CD; and the triangle ACL the ſame multiple of the triangle ACD. The conſtruction being thus finiſhed let us proceed to the demonſtration.

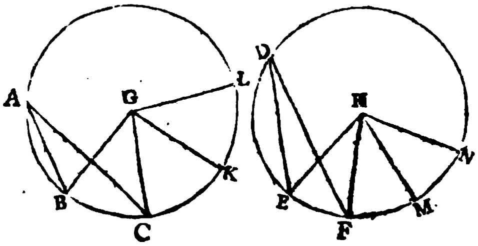


Now I ſay that whenever CH is leſs than CL; the triangle AHC is leſs than the triangle ACL; becauſe I may cut off from CL a part equal to CH; and then joining the extremity of that line to the point A; there will be a triangle, equal to ACH within the triangle ACL; therefore when CH is leſs than CL, the triangle AHC will always be leſs than the triangle ACL; and if CH be equal to CL, then (by 38. 1.) the triangle AHC is equal to ACL: and laſtly if CH be greater than CL, it may in like manner be proved that the triangle AHC is greater than the triangle ACL: and all this will be true according to any multiplication whatſoever. And if for the ſake of diſtinctneſs we call BC the firſt; CD the ſecond; the triangle ABC the third; and the triangle ACD the fourth; we have this general concluſion; if two ſuch triangles be between the ſame parallel lines; and if you take any equimultiples whatever of the firſt and third; and alſo any equimultiples whatever of the ſecond and fourth; whenever the multiple of the firſt exceeds the multiple of the ſecond the multiple of the third will always exceed the multiple of the fourth: and when equal, equal: and when leſs, leſs. Which was to be demonſtrated.

The reader is now to proceed by himſelf and prove the ſame thing of the equimultiples of BC and of the parallelogram CE; and the equimultiples of CD and of the parallelogram CF; that is, that whenever the multiple of BC exceeds the multiple of CD; then the multiple of the parallelogram EC will exceed the multiple of the parallelogram CF: and when equal, equal: and when leſs, leſs.

Here our reaſoning ſeems to become ſomewhat more complicated than in the firſt four books: but if we attend carefully to the meaning of the terms, we ſhall find very little difference: for the multiple is compoſed by a repetition of the *ſame* magnitude; which may therefore be conſidered only as a compariſon of two diſtinct things: the term *equimultiples* implies only a diviſion of the magnitudes into the ſame number of equal parts: and even in this laſt compariſon, we have only two magnitudes to compare with one another, as to their being greater; equal; or leſs. So that if the ſtudent but chuſes to prepare himſelf properly, he ſtands upon the ſame firm ground here, as in the firſt four books: for the magnitudes are the ſame, only the mode of compariſon is new: Which muſt be made familiar by particular inſtances. Therefore let us ſuppoſe two equal circles ABC, DEF; and BGC, EHF angles at their centers, ſtanding upon the circumferences BC, EF: I ſay, if you take equimultiples of the circumference BC and of the angle BGC at a venture; and alſo any equimultiples of the circumference EF and of the angle EHF : if the multiple of BC exceed the multiple of EF ; I ſay the multiple of the angle BGC will always exceed the multiple of the angle EHF ; and if equal, equal : and if leſs, leſs.

For take any multiple of the circumference BC, by placing or applying in the circle ſtraight lines, equal to a ſtraight line joining B and C ; and thus making the circumferences CK, KL equal to the circumference BC (by 28. 3.) : and thus BL may repreſent any multiple of the circumference BC : but becauſe the circumferences BC, CK, KL are equal to one another, (by 27. 3.) the angles BGC, CGK, KGL are equal to one another ; therefore the angle BGL is a multiple of the angle BGC ; but moreover it is the ſame multiple of this angle, that the circumference BL is of BC ; therefore the circumference BL and the angle BGL are equimultiples of the circumference BC and of the angle BGC : And in the ſame manner we may ſhew that the circumference EN and the angle EHN are equimultiples of the circumference EF and of the angle EHF : now ſuppoſing as in the laſt inſtance, the circumference BC to be the firſt ; the circumference EF the ſecond ; the angle BGC the third ; the angle EHF the fourth ; I ſay that whenever the circumference BL, the multiple of the firſt, exceeds the circumference EN, the multiple of the ſecond ; then the angle BGL the multiple of the third, will exceed the angle EHN the multiple of the fourth ; for, becauſe BL exceeds EN, I may take of from BL a part equal to EN ; by placing in the circle a ſtraight line equal to the ſtraight line joining the points E and N ; and can thus prove that a part of the angle BGL is equal to EHN : and alſo if BL be equal to EN the angle BGL is equal to the angle EHN ; and in the ſame manner it may be proved that if the multiple BL be leſs than EN, then the angle BGL is leſs than the angle EHN ; and that all this will happen according to any multiplication whatever &c.



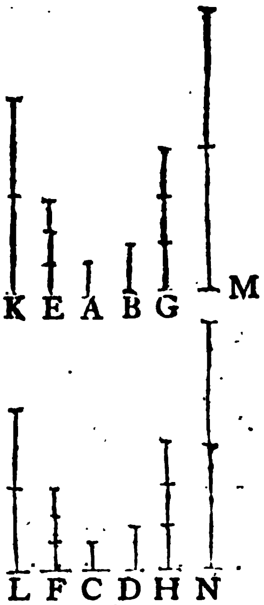
But here a difficulty ſeems to occur, becauſe the multiple may be ſo taken that BG and GL will become one ſtraight line. Alſo in taking a multiple of a ſtraight line; there is obviouſly no bounds ſet to the operation; which there ſeems to be in taking a multiple of a circumference; though this may be rendered indefinite by deſcribing as many equal circles as we pleaſe. Now what I take to be Euclid's meaning, as being moſt conſiſtent with the common notion of an angle, is this, that in comparing the multiples of the circumferences, all the ſemicircumferences are to be neglected; and in comparing the multiples of the angles at the center; we are only to conſider, that part of the angle which exceeds two right angles; which makes the conſtruction and figure here given general. I would recommend it to the ſtudent, to demonſtrate the ſame thing, of the multiples of the circumferences and ſectors of equal circles compared together; as alſo, of the multiples of the angles at the center and ſectors.

CHAP. V.

The ſame ſubject continued.

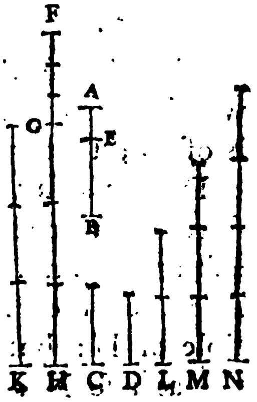
LET there be four magnitudes A, B, C, D; and let E, F be any equimultiples of A and C the firſt and third: and G and H any other equimultiples of B and D the ſecond and fourth: And let any equimultiples of A and C; and any equimultiples of B and D be ſo particularly connected with one another; that whenever the multiple of A exceeds the multiple of B; the multiple of C will always exceed the multiple of D; and when equal, equal: and when leſs, leſs: I ſay, that if any equimultiples of E and F be taken; and any whatever of G and H; it will always happen that when the multiple of E exceeds the multiple of G; the multiple of F will exceed the multiple of H; and when equal, equal; and when leſs, leſs. That is take K and L any equimultiples whatever of E and F; and alſo M and N any equimultiples whatever of G and H; I ſay that when K exceeds M; L will always exceed N; and when equal, equal; and when leſs, leſs.

THE



THE DEMONSRATION. Becauſe E and F are equimultiples of A and C; and K and L equimultiples of E and F; therefore, by the third propoſition of the fifth book, K and L are equimultiples of A and C: Again becauſe G and H are equimultiples of B and D; and M and N of G and H; therefore (by 3. 5.) M and N are equimultiples of B and D: but *the ſuppoſition is,* that if there be any equimultiples of A and C; and any whatever of B and D; if the multiple of A exceed the multiple of B; than the multiple of C will always exceed the multiple of D; and when equal, equal: and when leſs, leſs; therefore becauſe it has been demonſtrated that K and L are equimultiples of A and C and M and N of B and D; it follows that when K exceeds M; L will exceed N; and when equal, equal: and when leſs; leſs. Which was to be demonſtrated.

Let AB, C and D be three ſuch ſtraight lines; that any equimultiples whatever of AB and C compared with any multiple of D; may be thus related; that whenever the multiple of AB exceeds the multiple of D; the multiple of C ſhall alſo exceed the multiple of D; and when equal, equal: and when leſs, leſs: I ſay if this be the caſe AB is equal to C. For if not, one of them muſt be greater, which ſuppoſe to be AB. Cut off a part from AB equal to C; and let this be BE: multiply AE their difference until it exceed D: and let this multiple be FG; therefore FG is greater than D: make GH the ſame multiple of EB; that FG is of AE; therefore (by 1. 5.) FH is alſo the ſame multiple of AB: make K the ſame multiple of C; therefore FH and K are equimultiples of AB and C. Make L the double of D, and ſo on, until we come to the multiple which is the next greater than K: let this be N: M therefore is not greater than



K:

K: but K is equal to GH; becauſe EB is equal to C; therefore M is not greater than GH; but FG is greater than D (by conſt.) wherefore FH is greater than M and D together, that is N; therefore FH is greater than N; and K is not greater than N: and FH and K are equimultiples of AB and C and N is a multiple of D; but by the ſuppoſition any equimultiples of AB and C were ſo related to any multiple of D; that whenever the multiple of AB exceeds the multiple of D; the multiple of C alſo exceeds the multiple of D; but the contrary has been proved when we ſuppoſe AB unequal to C; therefore the multiples being according to the ſuppoſition AB cannot be unequal to C. Which was to be demonſtrated.

Let there be ſix magnitudes, or rather ſtraight lines A, B, C, D, E, F whoſe multiples are thus related: ſuppoſe any equimultiples of A and C to be taken, and alſo any equimultiples whatever of B and D; and farther whenever the multiple of A exceeds the multiple of B; let the multiple of C always exceed the multiple of D: and again let there be ſome equimultiples of C and E and alſo of D and F ſuch that the multiple of C exceeds the multiple of D; but the multiple of E does not exceed the multiple of F: I ſay that there are ſome equimultiples of A and E; and ſome of B and F ſuch that the multiple of A exceeds the multiple of B; when the multiple of E does not exceed the multiple of F.

Suppoſe G and H to be ſuch equimultiples of C and E; and K and L ſuch equimultiples of D and F; that G exceeds K but H does not exceed L: now becauſe by the ſuppoſition G and K are given, I can find what number of times C and D are contained in them by meaſuring them, that is by cutting off parts equal to C and D. Take M the ſame multiple of A that G is of C; that is, which H is of E: again take N the ſame multiple of B which K is of D; that is, which L is of F.

DEMONSTRATION. Now by the ſuppoſition A, B, C, D are ſo related to one another that any equimultiples of A and C; and alſo any equimultiples of B and D being taken; if the multiple of A exceed the multiple of B the multiple of C exceeds the multiple of D: but M and G are (by conſt.) equimultiples of A and C; as alſo N and K of B and D; therefore if M exceed N; G

exceeds

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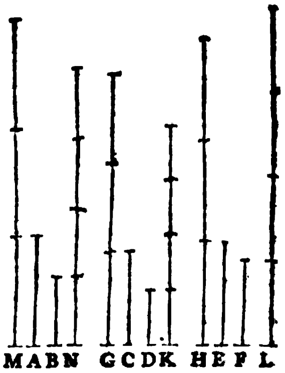
exceeds K: but G does exceed K (by ſupp.); therefore M exceeds N: but H does not exceed L (by ſupp.); wherefore M exceeds N but H does not exceed L; and M and H are equimultiples of A and E; and N and L of B and F. Wherefore &c. Which was to be demonſtrated.

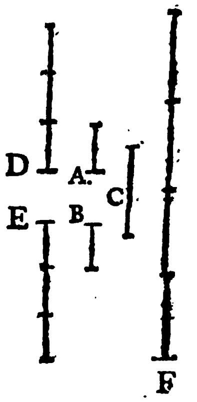
If D and E (ſee the next figure) be ſuch equimultiples of A and B; and F ſuch a multiple of C, that D exceeds F but E does not exceed F; then I ſay that A is greater than B.



For becauſe D exceeds F and E does not exceed F; therefore of conſequence D exceeds E: but becauſe D and E are equimultiples of A and B and D exceeds E, therefore any part of D will exceed the ſame part of E; that is, if one of the magnitudes be greater than another, the third &c. part of the one will exceed the third &c. part of the other; therefore A is greater than B.

Again if F be ſuch a multiple of C; and D and E ſuch equimultiples of A and B, that F exceeds E but F does not exceed D; then, I ſay that B is leſs than A.

For, becauſe F exceeds E but F does not exceed D; therefore E is leſs than D; therefore any part of E will be leſs than the ſame part of D; but A and B are the ſame parts of D and E therefore B is leſs than A. Which was to be demonſtrated.



CHAP. VI.

Containing an explanation of the remaining definitions.

IF what has been ſaid, in this diſſertation, be properly attended to, there will be little difficulty in underſtanding the meaning of the third definition; which has been the occaſion of ſo much trouble to the commentators; and which they have been ſo very unſucceſsful.

unſucceſsful in their attempts to explain. For by proceeding according to the ſpecimen here given, it is very obvious that all the properties of magnitudes, which are mentioned in the fifth book, may be demonſtrated from the two firſt definitions, without any mention of the term ratio. But to ſave the trouble of ſuch circumlocutions; our author gives particular names to the different ſuppoſitions; and this is the true origin of the following definitions, from the ſecond. Now ſays he, in the third, any certain mutual habitude of magnitudes of the ſame kind, conſidered according to this *quantuplicity* or multiplicity; or according to this relation of parts and multiples, I call *ratio*. And I believe I may ſay that this definition has never been rightly underſtood ſince the greek became a dead language. *Barrow* calls it a metaphyſical definition; but it appears from this, that it is a mathematical one. He ſays that ſuch definition was equally neceſſary in the ſeventh book, when ſpeaking of the ratio of numbers, but it appears from this explanation, that ſuch a definition could have no place there.

When any equimultiples of the firſt and third, and any equimultiples whatever of the ſecond and fourth, have this relation to one another, that when the multiple of the firſt exceeds the multiple of the ſecond; the multiple of the third always exceeds the multiple of the fourth; or when equal, equal; or when leſs, leſs: then he ſays the firſt has the ſame ratio to the ſecond which the third has to the fourth: But if the multiple of the firſt exceed the multiple of the ſecond, when the multiple of the third does not exceed the multiple of the fourth; he then ſays that the firſt has a greater ratio to the ſecond than the third has to the fourth. The ſixth, eighth, and ninth definitions are ſufficiently obvious. When three magnitudes are proportionals; the firſt is ſaid to have to the third the duplicate ratio, of that which it has to the ſecond. But this definition and the next I ſhall have occaſion to explain afterwards.

When we ſay that the firſt has the ſame ratio to the ſecond which the third has to the fourth; the firſt and third are the antecedents; and the ſecond and fourth are called the conſequents. Now the antecedents are called homologous terms, or terms of like ratio; and ſo are the conſequents alſo. This diſtinction is neceſſary for ſeveral reaſons; in the following definitions, and in fixing the equal angles of ſimilar triangles &c, and therefore ought to be particularly attended to.

The following definitions will be ſufficiently underſtood, when the particular propoſitions are read, wherein it is proved that the magnitudes will have the ſame ratio, the firſt to the ſecond and the third to the fourth, after they have undergone the changes, mentioned in theſe definitions. I know there is a way of explaining theſe definitions by numbers, which is nothing to the purpoſe; for Euclid is not ſpeaking of numbers at preſent but of magnitudes. However as this ſupplies the place of all the knowledge which this book contains to many, who are not ſo ſcrupulous as to require a demonſtration; nor ſo attentive as to conſider whether they are talking about ſomething or nothing, to gratify ſuch indolent readers, I have preſented them with the following ſcheme, ſerved up in the true French taſte.

$$15:5::30:10\begin{cases}\text{Alt.} & 15:30::5:10\\ \text{Inv.} & 5:15::10:30\\ \text{Comp.} & \begin{cases}15+5:5::30+10:10\\ 20:5::40:10\end{cases}\\ \text{Div.} & \begin{cases}15-5:5::30-10:10\\ 10:5::20:10\end{cases}\\ \text{Conv.} & 15:15-5::30:30-10\\ & 15:10::30:20\end{cases}$$

$$\begin{array}{c}\text{By equal.}\\ \text{ord.}\end{array}\begin{cases}3.\ 6.18\\ 5.10.30\end{cases}\qquad \begin{array}{c}\hline 3:18::5:30\end{array}$$

$$\begin{array}{c}\text{By equal.}\\ \text{pert.}\end{array}\begin{cases}7.21.42\\ 8.16.48\end{cases}\qquad \begin{array}{c}\hline 7:42::8:48\end{array}$$

I hope, from what has been ſaid, the reader underſtands the meaning of theſe definitions. The triangle, rectangle and circle have been conſidered as inſtruments of inveſtigation; but this doctrine of *ratios* is to be regarded as a new mode of compariſon, very extenſive, in its conſequences, giving us a wonderful command over the magnitudes, which we have already conſidered, by diſcovering a great many properties far beyond the reach of the former method of compariſon. And although a great deal of pains has been taken to render our ideas upon this ſubject, confuſed, indiſtinct and nothing; yet it is wonderful, when properly examined, what a ſimplicity there is in the principles upon which this method of compariſon is founded; the conſtructions are all performed (by 3. 1.) and requires no ſuch *apparatus* as is neceſſary even to prove one line to be equal to, or ſhorter than another, when we cannot make uſe of the circle.

CHAP. VII.

Of the arrangement of the propoſitions.

THE reader who has been accuſtomed to have his head filled with numerical ideas as explanatory of the propoſitions contained in this book, will perhaps be ſurprized to hear me affirm that he has taken a great deal of pains to fill his head with abſurdities. For I am perſuaded that if a ſcheme were to be thought of, for depriving a man of his reaſon, ſtill keeping up an impreſſion upon his mind that he was a rational creature, there could not be a more effectual method, than to ſet him upon reconciling the demonſtrations of this book to numerical ideas. Indeed the twiſting ropes of ſand would be a rational employment when compared with this.

All numerical reaſoning proceeds upon the ſuppoſition, that the unit is the ſame. But Euclid is not yet prepared for this confinement, which I ſhall prove particularly, upon the ſeventh book. He does not carry on his demonſtrations in the firſt four books, upon the ſuppoſition that the ſides of his triangles, or parallelograms or the radius of his circle, are three or four feet long; or as having a reference to any kind of meaſure. Nor is his reaſoning in this book leſs general. His *ratio* is a mutual habitude of magnitudes according to quantuplicity, or according to the doctrine of parts and multiples. The queſtion is not; is the part *one*; but is it a fixt magnitude? and farther, if you mean to reaſon upon it according to this doctrine of *ratios*; is it a magnitude that you can multiply? that is, that you can double; for the whole operation conſiſts only of doubling. It would not be more abſurd, to ſuppoſe that the nature of things, or human imagination does not allow of any triangle but an equilateral one, and nevertheleſs to try to prove that the ſquare of one ſide of a triangle might be equal to the ſquares of the other two; than to attempt the proof of ſome of the ſimpleſt properties of triangles, parallelograms and circles upon the ſuppoſition that all magnitudes can be expreſt by numbers.

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It is to be hoped what has been ſaid will be ſufficient to induce the reader to pay a proper attention to the ſtructure of the demonſtrations made uſe of in this book, which have a wonderful elegancy and force even when taken ſingly; but more particularly when the arrangement of the whole is conſidered: for the better underſtanding of which I would adviſe the reader to divide the book into four portions. The nature of *ratios* ſhews the propriety and neceſſity of introducing the properties of equimultiples firſt of all; and although the fourth propoſition is not abſolutely confined to its place; yet as it depends upon the third, it would be difficult to give it a more advantageous poſition. The firſt ſix propoſitions therefore will make the firſt portion. Again, we may have a very diſtinct notion of this *ratio* of magnitudes without knowing what the magnitudes themſelves are directly: Here therefore ſome rule or teſt is neceſſary for enabling us to get at the common relations of greater, equal and leſs, when we have only this *ratio* to guide us. The ſtudent is therefore to conſider the ſeventh, eighth, ninth and tenth propoſitions, as making a diſtinct ſection, and as introduced for the very purpoſe of ſettling this point.

It might appear to a ſuperficial obſerver, who had juſt come from proving that if four magnitudes are proportionals, they will be proportionals by inverſion; that a ſimilar method of reaſoning might be applied to ſhew that they will be proportionals by alternation. But it happens here, that the equimultiples, of the firſt and third, of the magnitudes which we want to prove to be proportionals; are not equimultiples of the ſecond and fourth of thoſe magnitudes which are proportionals by the ſuppoſition; but of the firſt and ſecond; which deſtroys all proſpect of ſucceeding according to that plan. And indeed this demonſtration is no eaſy matter, for all the propoſitions from the tenth to the ſeventeenth may be conſidered as introduced to demonſtrate; that magnitudes will be proportionals by alternation. And this will make up the third portion according to the diviſion above mentioned. And the remaining part of the book may be conſidered under one head, in which is proved that magnitudes will be proportionals; by compoſition diviſion, converſion, and by the two methods of reaſoning by *equal-diſtances*. And thus I think a very diſtinct and comprehenſive view of the ſubject of this book may be taken, and ſuch an one as ſeems likely to fix it in the memory.

CHAP. VIII.

Containing ſome remarks on the demonſtrations.

AGREEABLE to the rule already laid down, inſtead of remarks upon particular demonſtrations, I ſhall reduce what I have to ſay under diſtinct heads, which the reader may apply as he goes along; but firſt it ſeems neceſſary to be a little particular, as to the conſtruction of the thirteenth propoſition. Indeed there is nothing which ſeems to be ſo prevailing an error, with regard to this book, as a neglect of the conſtructions. And yet this is a thing which Euclid is, every where, more particularly attentive to. He ſometimes leaves his reader to draw the conſequence from a conſtruction, when it is ſufficiently obvious, and cannot be done ſhortly and elegantly; an inſtance of which we have in the firſt propoſition of the third book; where there are two conſtructions, but he reaſons directly, only from one of them, and then joins both their conſequences in his concluſion: for the demonſtration only proves that the center muſt be in the perpendicular; for no abſurdity will follow from the reaſoning, by ſuppoſing the point G to be any where in the line CD; not even if we ſuppoſe it to be at the circumference; for it will not then be abſurd to ſuppoſe the angle GDB equal to CDB; we have therefore Euclid's authority for drawing conſequences in this manner; and I have not the leaſt doubt of this demonſtration being the genuine one given by Euclid; becauſe to have made it quite regular the two conſtructions muſt have been ſeparated; and then it would not have been one, but two diſtinct propoſitions; in the firſt of which it was to be proved that the center muſt be in the perpendicular; and in the ſecond, that this perpendicular cut in halves would find the center; which is a much greater formality than the nature of the problem requires, although the ſtudent ought to examine it in this manner. But in this thirteenth propoſition the caſe ſeems to be very different, and conſequences are drawn from a conſtruction which I believe

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believe it will be found very difficult to perform. *Simſon* in his note upon this propoſition ſays; " In Commandine's, Brigg's and " Gregory's tranſlations, at the beginning of this demonſtration, " it is ſaid, *and the multiple of C is greater than the multiple of D; " but the multiple of E is not greater than the multiple of F*; which " words are a literal tranſlation from the greek: But the ſenſe evi- " dently requires that it be read, ſo that the multiple of C be " greater than the multiple of D; but the multiple of E be not " greater than the multiple of F." Now I am ſo much of a contrary opinion, that it appears to me that this change deſtroys every veſtige of the true conſtruction of this propoſition. My notion is that the proper equimultiples are given me by the hypotheſis; and the multiples and magnitudes being given, I can meaſure the one by the other; and conſequently know how many times the magnitudes are taken; from which the conſtruction is obvious: but if I have nothing to direct me but the magnitudes themſelves; concerning which alſo there is no ſuppoſition of greater or leſs, I ſhould be glad to know in ſuch circumſtances, how this conſtruction is to be performed; I know it is ſaid *let them be taken*, which if I were to add my conjectures after the manner of ſome commentators, I would ſuſpect to be the correction of ſome haſty editor who did not underſtand the demonſtration.

In proving four magnitudes to have the ſame ratio, the firſt to the ſecond which the third has to the fourth; it is neceſſary that the equimultiples of the firſt and third be *any whatever*; and alſo that the equimultiples of the ſecond and fourth be taken at a venture. Now it is obvious that if they be taken as often as one determinate magnitude contains another, they cannot be *any whatever*; neither is the ſum of two multiples, each of which has been taken at a venture, to be conſidered as any multiple whatever; becauſe if in one caſe it ſhould happen to be taken three times, and in the other five times; theſe multiples added together, would not *happen* to be eight times that magnitude, but muſt really be eight times the magnitude, not accidentally but conſequentially: again when a multiple of a magnitude is taken, and then a multiple of that multiple as in the third propoſition; this laſt is not *any multiple whatever* of the firſt magnitude; for inſtance the firſt happens to be taken three times, and this multiple again five times; now the laſt multiple does not *happen* to be fifteen times the firſt, magnitude, but muſt be really ſo. And this will be ſufficient to direct the ſtudent to the proper uſe of the phraſe *any whatever*.

Simſon's remarks upon the eighteenth propoſition are very ingenious; for the poſſibility of the fourth proportional is not to be taken for granted: becauſe this is a very different notion from that which ſuperficial people are apt to confound with it: for when two magnitudes are made the ſubject of contemplation, their *exiſtence* is a part of the ſuppoſition; and if they do exiſt, it muſt be under one or other of theſe forms; A muſt be equal to B, or greater, or leſs; and if I prove that A cannot be greater than B nor leſs, I certainly demonſtrate that A is equal to B. But it is very different where the *exiſtence* of the magnitude may be called in queſtion, which is very reaſonably done in the preſent inſtance; for it may be ſaid to Euclid you have confined yourſelf ſo much by your definition of *the ſame ratio*, that a fourth proportional may be really impoſſible according to your definition; but when you have once ſhewn the poſſibility of it, I will then admit your reaſoning. It is very certain, that we have ſufficient principles, before this propoſition, for finding a fourth proportional to any three given ſtraight lines, which would make the demonſtration unexceptionable as we have it at preſent; by proceeding thus; to AB, BE and CD find a fourth proportional, which muſt be greater, equal or leſs than FD &c; but this would be mixing things together, which our author certainly intended to keep ſeparate. But to conclude this chapter, whoever has a proper notion of a part and multiple; and knows what is required to prove that four magnitudes have the ſame ratio the firſt to the ſecond which the third has to the fourth, can never be miſtaken in forming a judgement of the demonſtrations contained in this book. Euclid conſtantly ſuppoſes his magnitudes to be ſtraight lines as is obvious from his conſtructions and demonſtrations; ſo that there is no occaſion to ſuppoſe the magnitudes in any propoſition to be of the ſame kind; and indeed thoſe critics who have made ſuch alterations, if they had underſtood their buſineſs, ought at leaſt to have changed the enunciations of half the propoſitions in the book.

CHAP.

CHAP. IX.

Of the ſubject and arrangement of the ſixth book.

OUR author having laid down this general doctrine of parts and multiples proceeds to apply it to the inveſtigation of ſuch properties of the triangle, rectangle and circle as could not be obtained by the former method of compariſon. For what is delivered in the fifth book itſelf cannot, ſtrictly ſpeaking, be conſidered as properties of triangles, parallelograms &c. but only as properties of parts and multiples of ſuch magnitudes as come under the two firſt definitions. But in this book he treats of the triangle as ſuch, and ſhews when triangles have the ſame altitude, they have the ſame ratio to one another, as their baſes. And here the thoughtleſs reader will do well to obſerve; that every triangle may have its altitude expreſt by three different ſtraight lines, as each ſide can be the baſe: for the altitude of a triangle is a relative term, and has no meaning until the baſe is fixt. From this he demonſtrates that if the two ſides of a triangle be cut by a ſtraight line parallel to the third ſide, they will be cut in the ſame proportion; this is the fundamental principle which runs through the whole of this book; and ought for this reaſon to be examined in every point of view. For after the reader has conſidered this property according to its moſt obvious acceptation, he will find it more general than he ſuſpected, by making it a property of two indefinite ſtraight lines interſecting one another, and cut in any manner by two parallel lines.

It has been mentioned already that the triangle is the great inſtrument of geometrical inveſtigation, and Euclid's firſt object in this book is to lay down thoſe principles upon which the ſimilarity of triangles depends; and by this acquiſition the power of the ſcience becomes aſtoniſhingly great, as it is difficult to ſay what cannot be done by it that is really practicable; the judicious reader therefore will examine the firſt eight propoſitions with that attention which the importance of the ſubject requires. The five following propoſitions teach how to divide a ſtraight line into any number of equal parts; to cut a given line in the ſame proportion, as another has been cut; to find a third, fourth and mean proportional; which being practical principles, ſhould be made habitual. The next four propoſitions are all upon the ſame ſubject, and that a very curious one, namely to fix the equality of the parallelograms and triangles from the ratio of the ſides and converſly. And here the indolent reader, ſhould be required to ſhew what uſe is made of the equality of the angles, or where the demonſtration would fail if that circumſtance were omitted. The nineteenth and twentieth propoſitions are of great importance and ſhould be carefully ſtudied: it will aſſiſt a beginner very much to conſider them as problems, omitting the term duplicate ratio entirely: Thus the ninteenth may be expreſt in this manner; two ſimilar triangles being given, to find two ſtraight lines which ſhall have the ſame ratio to one another which the triangles have. And BC and BG will be the two ſtraight lines required. The reader may know whether he underſtands the propoſition, by taking a third proportional to EF and BC, and compleating the conſtruction and demonſtration. The twentieth may be worded thus; to find two ſtraight lines which ſhall have the ſame ratio to one another which two ſimilar polygons have; and then to prove that the ſame two ſtraight lines, will have the ſame ratio to one another which the ſimilar triangles, into which the polygons are divided, have to one another. And it will alſo be convenient, according to the ſame plan, to conſider the twenty third as a problem; which may be expreſt in this manner; to find two ſtraight lines which ſhall have the ſame ratio to one another which two equiangular parallelograms have; without making any uſe of the phraſe *ratio compounded of ratios*; and for a proper explanation of this term the reader may conſult *Simſon's* edition. I would propoſe it to the reader to examine theſe propoſitions firſt as problems, becauſe this will fix his attention more to the particular ſteps of the conſtruction and the conſequences drawn from them. And alſo the extent and importance of theſe very beautiful theorems is more to be ſeen by conſidering them in this practical point of view. For by the twentieth propoſition, figures may be increaſed or diminiſhed in any proportion. For inſtance if I want to make a figure the fifth part of a given figure; I take the fifth part of one of the ſides of the given figure; and find a mean proportional between the line and its fifth part, upon which if a ſimilar and ſimilarly ſituated figure be deſcribed, it will be the fifth part of the given figure. Likewiſe by this propoſition the forty ſeventh of the firſt book, may be extended to any ſimilar rectilineal figures.

The twenty third is often quoted to prove, that, if there be four lines which have the ſame ratio; and alſo other four, the rectangles contained by the antecedents and conſequents will have the ſame ratio, the firſt to the ſecond which the third has to the fourth. This the reader ſhould examine, which he may eaſily do by the aſſiſtance of the note below. *

The ſubject and arrangement of what remains is ſo obvious, that it would be paying the reader but a poor complement to ſuppoſe that he ſtood in need of any farther information, eſpecially if he read what *Simſon* has ſaid upon the 28, and 29 propoſitions.

I have ſaid nothing of corollaries; and it is doubtful in what ſenſe Euclid would have them be underſtood. In the firſt four books, they are all ſome circumſtances which are in fact demonſtrated in the propoſition, but which, not being expreſt in words, the reader might without this notice have overlooked; but he ſeems afterwards not to confine himſelf to this ſenſe, as will be obvious to the reader.

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* See prop. 10. B, 2. Ham. Con. Sect.

$$\begin{array}{l}\text{EP. PD. PF. PH}\\ \text{BM. MA. BM. MC.}\end{array}\left\{\begin{array}{l}\text{K. L. M.}\\ \text{N. O. P.}\end{array}\right.$$

Then (by equal.) K : M :: N : P.

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THE ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

1. A Part is a magnitude of a magnitude, the leſs of the greater, when it meaſures the greater. 2. But a multiple, the greater of the leſs, when it is meaſured by the leſs.

3. Ratio is a certain habitude of magnitudes of the ſame kind to one another, which *is only* as to quantity.

4. Magnitudes are ſaid to have a ratio to one another, which may be multiplied ſo as to exceed one another.

5. Magnitudes are ſaid to be in the ſame ratio; the firſt to the ſecond and the third to the fourth; when the equimultiples of the firſt and third, according to any multiplication, are at the ſame time leſs, or at the ſame time equal, or are at the ſame time greater than each of the equimultiples of the ſecond and fourth, compared with one another. 6. And let magnitudes, having the ſame ratio, be called proportionals. 7. But when of equimultiples; the multiple of

the firſt exceeds the multiple of the ſecond, but the multiple of the third does not exceed the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third to the fourth. 8. And proportion is a ſimilitude of ratios. 9. But proportion conſiſts in three terms at leaſt. 10. And when three magnitudes are proportionals; the firſt is ſaid to have to the third a duplicate ratio of that which it has to the ſecond. 11. And when four magnitudes are proportionals; the firſt is ſaid to have to the fourth a triplicate ratio of that which it has to the ſecond; and always in order one more, as long as the proportion continues.

12. The leading magnitudes are ſaid to be of like ratio with the leading magnitudes; and thoſe that follow *are ſaid to be of like ratio* with thoſe that follow. Or. The antecedents are ſaid to be homologous magnitudes to the antecedents; and the conſequents to the conſequents.

13. Alternate ratio is the taking of the antecedent to the antecedent; and of the conſequent to the conſequent.

14. Inverſe ratio is the taking of the conſequent as an antecedent to the antecedent as a conſequent.

15. The compoſition of a ratio is the taking of the antecedent together with the conſequent, as one, to the conſequent. 16. But the diviſion of a ratio is the taking of the exceſs, by which the antecedent exceeds the conſequent, to the conſequent itſelf.

17. The converſion of a ratio is the taking of the antecedent, to the exceſs, by which the antecedent exceeds the conſequent.

18, A ratio of equality is, there being ſeveral magnitudes, and others equal to them in multitude, and in the ſame proportion taken two by two, when it is, as in the firſt magnitudes; the firſt to the laſt, ſo in the ſecond magnitudes the firſt to the laſt. or otherwiſe. The taking of the extremes by a ſubſtraction of the middle *terms*.

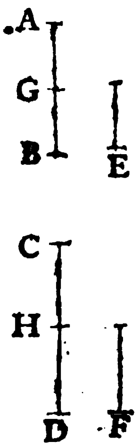
19. Ordinate proportion is, when it is as the antecedent to the conſequent, ſo is the antecedent to the conſequent; and it is as the conſequent to ſome other, ſo is the conſequent to ſome other. 20. But perturbate proportion is, when there being three magnitudes and others equal to them in number, it is as in the firſt magnitudes the antecedent to the conſequent, ſo in the ſecond magni-

tudes the antecedent to the conſequent; but as in the firſt magnitudes the conſequent is to ſome other, ſo in the ſecond magnitudes is ſome other to the antecedent.

PROP. I.

If there be any number of magnitudes, equimultiples of an equal number of magnitudes, each of each; whatſoever multiple one of the magnitudes is of one, the ſame multiple will all be of all.

Let AB, CD be any number of magnitudes, equimultiples of an equal number of magnitudes E, F; each of each: I ſay that whatſoever multiple AB is of E, the ſame multiple will AB, CD *together* be of E, F *together*.



For becauſe AB is the ſame multiple of E that CD is of F; as many magnitudes as there are in AB equal to E, ſo many are there in CD equal to F; let AB be divided (by 3. 1.) into the magnitudes AG, GB equal to E; and CD into the *magnitudes* CH, HD equal to F: certainly the number of *parts* CH, HD will be equal to the number of *parts* AG, GB: and becauſe AG is equal to E, and CH *equal* to F, therefore AG, CH *together are* equal to E and F *together*; for the ſame reaſon GB is equal to E, and HD to F; therefore alſo GB, HD *together* are equal to E, F *together*: therefore as many *magnitudes* as there are in AB equal to E; ſo many are there in AB, CD *together* equal to E, F *together*: wherefore whatſoever multiple AB is of E, the ſame multiple will AB, CD *together* be of E, F *together*.

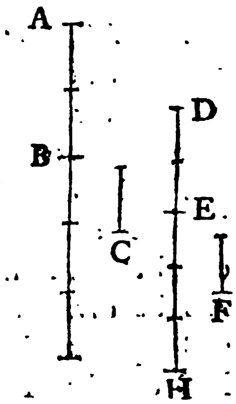
Wherefore if there be any number of magnitudes, equimultiples of an equal number of magnitudes, each of each; whatſoever multiple one of the magnitudes is of one, the ſame multiple will all be of all. Which was to be demonſtrated.

PROP. II.

If the firſt be the ſame multiple of the ſecond as the third is of the fourth; and if the fifth be the ſame multiple of the ſecond as the ſixth *is* of the fourth : alſo the firſt and fifth together will be the ſame multiple of the ſecond, as the third and ſixth together *is* of the fourth.

For let the firſt AB be the ſame multiple of the ſecond C as the third DE is of the fourth F; and let the fifth BG *be* the ſame multiple of the ſecond C as the ſixth EH *is* of the fourth F; I ſay that AG the firſt together with the fifth will be the ſame multiple of theſecond C as DH the third and ſixth *is* of F the fourth.

For becauſe AB is the ſame multiple of C that DE *is* of F; as many magnitudes as there are in AB equal to C, ſo many *will there be* in DE equal to F: certainly for the ſame reaſon alſo, as many *magnitudes* as there are in BG equal to C, ſo many are there alſo in EH equal to F; therefore as many as there are in the whole AG equal to C ſo many *are there* in the whole DH equal to F: wherefore whatſoever multiple AG is of C, the ſame multiple will DH be of F; therefore AG the firſt together with the fifth will be the ſame multiple of C the ſecond, that DA the third and ſixth *is* of F the fourth.



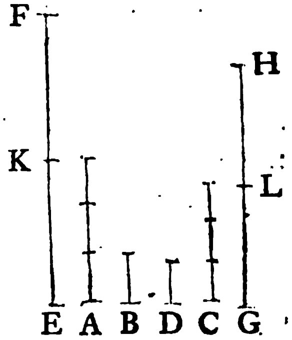
Wherefore if the firſt be the ſame multiple of the ſecond as the third *is* of the fourth; and if the fifth be the ſame multiple of the ſecond as the ſixth *is* of the fourth; alſo the firſt and fifth together will be the ſame multiple of the ſecond, as the third and ſixth together *is* of the fourth. Which was to be demonſtrated.

PROP. III.

If the firſt be the ſame multiple of the ſecond as the third *is* of the fourth; and equimultiples be taken of the firſt and third; alſo by equality, each of thoſe taken will be an equimultiple, of each, the one of the ſecond, and the other of the fourth.

For let the firſt A be the ſame multiple of the ſecond B, as the third C *is* of the fourth D; and let EF, GH be taken equimultiples of A *and* C: I ſay that EF is the ſame multiple of B, which GH *is* of D.

For becauſe EF is the ſame multiple of A (by conſt.) which GH *is* of C: therefore as many *magnitudes* as there are in EF equal to A, ſo many *are there* in GH equal to C: let EF be divided into magnitudes equal to A, viz. EK, KF; and GH into *magnitudes* equal to C, viz. GL, LH; the number of *parts* EK, KF will be equal to the number of *parts* GL, LH: And becauſe A is the ſame multiple of B, as C *is* of D; and EK is equal to A, and GL equal to C; therefore EK is the ſame multiple of B, which GL is of D. Certainly for the ſame reaſon alſo KF is the ſame multiple of B, which LH is of D: wherefore becauſe the firſt EK is the ſame multiple of the ſecond B, as the third GL *is* of the fourth D; and alſo the fifth KF is the ſame multiple of the ſecond B, which the ſixth LH *is* of the fourth D; therefore (by 2. 5.) EF the firſt together with the fifth is the ſame multiple of the ſecond B, which GH the third and ſixth *is* of the fourth D.



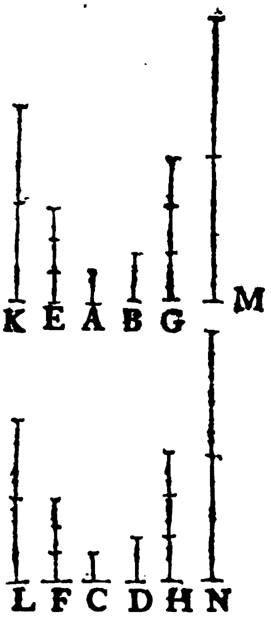
Wherefore if the firſt be the ſame multiple of the ſecond, as the third is of the fourth; and equimultiples be taken of the firſt and third: alſo by equality, each of thoſe taken will be an equimultiple of each, the one of the ſecond, and the other of the fourth. Which was to be demonſtrated.

PROP. IV.

If the firſt have the ſame ratio to the ſecond as the third to the fourth; alſo the equimultiples of the firſt and third, will have the ſame ratio to the equimultiples of the ſecond and fourth, according to any multiplication whatſoever; taken ſo as to anſwer each other.

For let the firſt A have the ſame ratio to the ſecond B, which the third C *has* to the fourth D; and let E *and* F be taken equimultiples of A *and* C; but let G and H *be taken* any equimultiples of B and D which may accidentally happen; I ſay that as E is to G ſo is F to H.

For let K *and* L be taken equimultiples of E *and* F; and M, N *any other* equimultiples of G *and* H *which may happen.*



And becauſe E is the ſame multiple of A which F is of C; and K *and* L have been taken equimultiples of E *and* F; therefore (by 3. 5.) K is the ſame multiple of A which L *is* of C; Certainly for the ſame reaſon M is the ſame multiple of B which N *is* of D: and becauſe (by ſupp.) A is to B as C is to D; and K *and* L have been taken equimultiples of A *and* C; and M *and* N [any other] equimultiples of B *and* D [which may accidentally happen]; therefore (by 5 def. 5.) if K exceed M alſo L exceeds N; and if equal, equal; and if leſs, leſs: and K *and* L are equimultiples of E *and* F; and M *and* N any other equimultiples, which may accidentally happen; of G *and* H; therefore (by 5 def. 5.) E is to G as F is to H.

Wherefore if the firſt have the ſame ratio to the ſecond as the third to the fourth; alſo the equimultiples of the firſt and third, will have the ſame ratio to the equimultiples of the ſecond and fourth; according to any multiplication whatſoever; taken ſo as to anſwer each other. Which was to be demonſtrated.

Cor. Wherefore becauſe it has been demonſtrated, that if K exceed M; L exceeds N, and if equal, equal: and if leſs, leſs. Certainly alſo if M exceed K, N exceeds L: and if equal, equal; and if leſs, leſs: and on this account it will be; as G is to E, ſo is H to F. Certainly from this it is manifeſt that if four magnitudes be proportionals, they will be inverſly proportionals; (by def. 14. 5.).

PROP. V.

If a magnitude be the ſame multiple of a magnitude, which a *magnitude* taken from *the one* is of *a magnitude* taken from *the other*; whatſoever multiple the whole is of the whole, alſo the remainder will be the ſame multiple of the remainder.

For let the magnitude AB be the ſame multiple of the magnitude CD, which AE the *magnitude* taken from *the one* is of the

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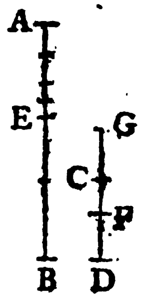
magnitude CF taken from *the other*: I ſay that whatſoever multiple the whole AB is of the whole CD, alſo EB the remainder will be the ſame multiple of FD the remainder.

For whatſoever multiple AE is of CF; let EB be the ſame multiple of CG.



And becauſe AE is the ſame multiple of CF (by the conſt. and 1. 5.) *which* AB is of GF; and AE is ſuppoſed *to be* the ſame multiple of CF *which* AB is of CD: therefore AB is the ſame multiple of either of the *lines* GF, CD; therefore (by com. not. 7.) GF is equal to CD; let CF *which is* common be taken away; therefore the remainder GC is equal to the remainder DF; and becauſe AE is the ſame multiple of CF which EB is of GC; and GC is equal to DF; therefore AE is the ſame multiple of CF which EB is of FD; but AE is ſuppoſed *to be* the ſame multiple of CF, which AB is of CD: therefore EB is the ſame multiple of FD *which* AB is of CD; therefore whatſoever multiple the whole AB is of the whole CD; alſo the remainder EB is the ſame multiple of the remainder FD.

Wherefore if a magnitude be the ſame multiple of a magnitude, which a *magnitude* taken from *the one* is of a magnitude taken from *the other*; whatſoever multiple the whole is of the whole, alſo the remainder will be the ſame multiple of the remainder. Which was to be demonſtrated.

PROP. VI.

If two magnitudes be equimultiples of two magnitudes; and ſome *magnitudes* equimultiples of the ſame be taken away; alſo the remainders are either equal to thoſe magnitudes, or equimultiples of them.

For let the two magnitudes AB, CD be equimultiples of the two magnitudes E, F; and let AG, CH the *magnitudes* taken away be equimultiples of the ſame E, F: I ſay that the remainders GB, HD are either equal to E, F or equimultiples of them.

For firſt let GB be equal to E; I ſay that HD is alſo equal to F: for put CK equal to F.

And

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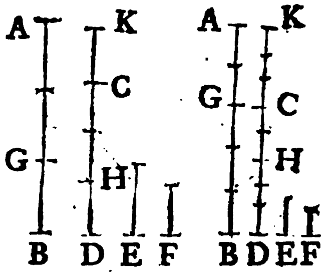


And becauſe AG is the ſame multiple of E as CH *is* of F; and GB *is* equal to E; and CK *equal* to F; therefore (by 2. 5.) AB is the ſame multiple of E *which* KH *is* of F; but AB is ſuppoſed *to be* the ſame multiple of E, *which* CD *is* of F; therefore KH is the ſame multiple of F, which CD *is* of F; wherefore becauſe each of the *lines* KH, CD *is* the ſame multiple of F; therefore (by com. not. 6.) KH is equal to CD; let CH *which is* common be taken away; therefore the remainder KC is equal to the remainder HD: but KC is equal to F; therefore HD is alſo equal to F; ſo that when GB is equal to E, HD will alſo be equal to F.

Certainly in the ſame manner we ſhall demonſtrate, that when GB is a multiple of E, that HD will be the ſame multiple of F.

Wherefore if two magnitudes be equimultiples of two magnitudes; and ſome *magnitudes* equimultiples of the ſame be taken away; alſo the remainders are either equal to thoſe magnitudes, or equimultiples of them. Which was to be demonſtrated.

PROP. VII.

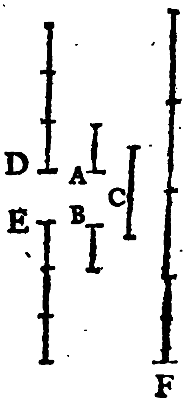
Equal *magnitudes* have the ſame ratio to the ſame *magnitude*; and the ſame *magnitude has the ſame ratio* to equal *magnitudes*.

Let A *and* B be equal magnitudes; and C any other magnitude which may accidentally happen; I ſay that each of the magnitudes A, B has the ſame ratio to C; and that C has the ſame ratio to each of the *magnitudes* A, B.

For let D, E be taken equimultiples of A, B; and let F be taken any other multiple of C which may accidentally happen.

And becauſe D is the ſame multiple of A, *which* E *is* of B; and A *is* (by ſupp.) equal to B; therefore D *is* equal to E; but F *is* any other multiple of C which may accidentally happen; wherefore if D exceed F, E alſo exceeds F; and if equal, equal; and if leſs, leſs: and D, E are equimultiples of A, B; and F any other multiple of C which may accidentally happen; therefore (by 5 def. 5.) it is as A *is* to C ſo is B to C.

I ſay alſo that C has the ſame ratio to each of the *magnitudes* A, B,



For the ſame things being conſtructed; in the ſame manner we ſhall demonſtrate that D is equal to E; and that F is any other *magnitude*; therefore if F exceed D; it alſo exceeds E: and if equal, equal: and if leſs, leſs: and F is a multiple of C; and D, E any other equimultiples of A and B which may accidentally happen; therefore (by 5. def. 5.) it is, as C to A ſo is C to B.

Therefore equal *magnitudes* have the ſame ratio to the ſame *magnitude*; and the ſame *magnitude has the ſame ratio* to equal *magnitudes*. Which was to be demonſtrated.

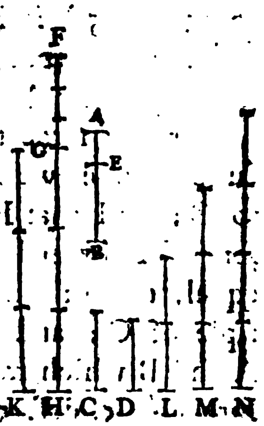
PROP. VIII.

Of unequal magnitudes; the greater has a greater ratio to the ſame *magnitude*, than the leſs *has*: and the ſame *magnitude* has a greater ratio to the leſs, than it has to the greater.

Let AB and C be unequal magnitudes; and let AB be greater than C; and let D be any other magnitude which may accidentally happen: I ſay that AB has a greater ratio to D than C has to D: And D has a greater ratio to C, than *it has* to AB.

For becauſe AB is greater than C; make BE equal to C: certainly (by 4. def. 5.) the leſſer of the *two* AE, EB being multiplied will at length be greater than D. Firſt let AE be leſs than EB; and let AE be multiplied until what is produced ſhall be greater than D: and let FG be the multiple of it, which is greater than D: and whatſoever multiple FG is of AE, let GH be the ſame multiple of EB; and K *the ſame multiple* of C: and let L be taken the double of D, and M triple; and more by one in order, until the multiple of D taken becomes the firſt greater than K: let it be taken: and let it be N four times D, the firſt greater than K.

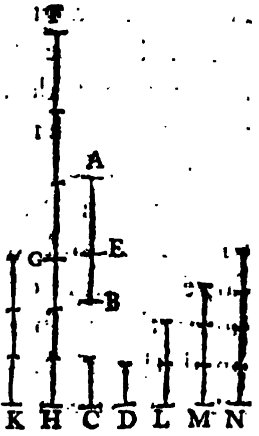
Wherefore becauſe K is the firſt leſs than N *or* [*N is the firſt multiple of D greater than K*]; therefore K is not leſs than M;



and becauſe FG is the ſame multiple of AE which GH *is* of EB; therefore (by 1. 5.) FG is the ſame multiple of AE which FH is of AB; and FG is the ſame multiple of AE which K is of C; therefore FH is the ſame multiple of AB, which K is of C; therefore FH and K are equimultiples of AB and C: Again becauſe GH is the ſame multiple of EB which K *is* of C; and EB is equal to C; therefore GH is equal to K: but K is not leſs than M; therefore neither is GH leſs than M; but (by conſt.) FG is greater than D; therefore the whole FH is greater than both D and M together; but D and M together are equal to N; therefore FH exceeds N, but K does not exceed N; and FH and K are equimultiples of AB and C; and N is any other multiple of D which may accidentally happen; therefore (by 7. def. 5.) AB has a greater ratio to D, than C has to D.

I ſay alſo, that D has a greater ratio to C, than D *has* to AB.

For the ſame things being conſtructed; in like manner we ſhall demonſtrate, that N exceeds K; but does not exceed FH; and N is a multiple of D; and FH *and* K [any other] equimultiples of AB and C [which may accidentally happen]; therefore (by 7. def. 5.) D has a greater ratio to C, than D *has* to AB.



But let AE be greater than EB; certainly EB the leſs being multiplied will at length (by 4. def. 5.) be greater than D: let it be multiplied; and let GH be the multiple of EB, greater than D; and whatſoever multiple GH is of EB; let FG be made the ſame multiple of AE; and K of C: Certainly in the ſame manner we ſhall demonſtrate, that FH *and* K are equimultiples of AB *and* C: and in like manner let N be taken, a multiple of D, the firſt greater than FG; ſo that again FG be not leſs than M; but GH is greater than D; therefore the whole FH exceeds D and M together, that is N; but K does not exceed N; ſince FG being greater than GH; that is K, does not exceed N; and in like manner following the *ſteps* above we finiſh the demonſtration.

Wherefore of unequal magnitudes, the greater has a greater ratio to the ſame *magnitude*, than the leſs *has*: and the ſame *magnitude* has a greater ratio to the leſs, than it has to the greater. Which was to be demonſtrated.

PROP. IX.

Magnitudes having the ſame ratio to the ſame *magnitude* are equal to one another; and thoſe are alſo equal to one another, to which the ſame *magnitude* has the ſame ratio.

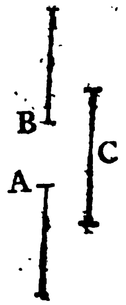
For let each of the *magnitudes* A, B have the ſame ratio to the ſame *magnitude* C: I ſay that A is equal to B.

For if not, (by 8. 5.) each of the *magnitudes* A, B could not have the ſame ratio to C: but it has (by ſupp.); therefore A is equal to B.

Again let C have the ſame ratio to each of the *magnitudes* A, B; I ſay that A is equal to B.

For if not; C could not (by 8. 5.) have the ſame ratio to each of the *magnitudes* A, B; but it has (by ſupp.); therefore A is equal to B.

Wherefore, *magnitudes* having the ſame ratio to the ſame *magnitude* are equal to one another; and thoſe are equal to one another to which the ſame *magnitude* has the ſame ratio. Which was to be demonſtrated.

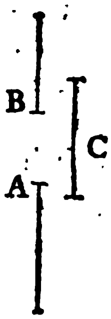


PROP. X.

Of *magnitudes* having ratio to the ſame *magnitude*, that is the greater which has the greater ratio: and that is the leſs to which the ſame has a greater ratio.

For let A have a greater ratio to C than B has to C: I ſay that A is greater than B.

For if not; either A is equal to B, or leſs; but A is not equal to B; for *then* each of the *magnitudes* A, B (by 7. 5.) has the ſame ratio to C; but (by ſupp.) *each of them* has not: therefore A is not equal to B; neither is A leſs than B; for (by 8. 5.) A would have a leſs ratio to C than B *would have* to C; but (by ſupp.) it has not; therefore A is not leſs than B; but it has been demonſtrated that neither is it equal: therefore A is greater than B.



Again let C have a greater ratio to B than C has to A: I ſay that B is leſs than A.

For if not; it is either equal, or greater: but B is not equal to A, for (by 7. 5.) C would have the ſame ratio to each of the *magnitudes* A, B: but (by ſupp.) it has not; therefore A is not equal to B: Neither is B greater than A; for (by 8. 5.) C would have a leſs ratio to B than to A; but (by ſupp.) it has not; therefore B is not greater than A; but it has been demonſtrated that neither *is* it equal; therefore B is leſs than A.

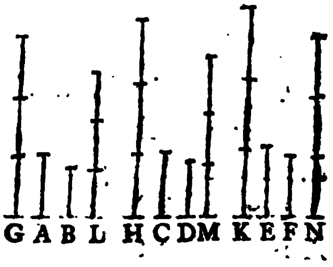
Wherefore of *magnitudes* having ratio to the ſame *magnitude*, that is the greater, which has the greater ratio: and that is the leſs to which the ſame has a greater ratio. Which was to be demonſtrated.

PROP. XI.

Ratios which are the ſame to the ſame ratio, are the ſame to one another.

For let the *ratios* be; as A to B ſo is C to D; and as C to D ſo is E to F; I ſay that it is, as A is to B ſo is E to F.

For let G, H, K be taken equimultiples of A, C, E; and let L, M, N *be taken* any other equimultiples of B, D, F which may accidentally happen.



And becauſe it is (by ſupp.) as A *is* to B ſo *is* C to D; and G *and* H have been taken equimultiples of A *and* C; and L, M any equimultiples of B, D which may accidentally happen; wherefore if G exceed L; H alſo exceeds M (by 5. def. 5.); and if equal, equal; and if leſs, leſs: Again, becauſe it is (by ſupp.), as C *is* to D ſo is E to F; and H, K have been taken equimultiples of C, E; and M, N any other equimultiples of D, F which may accidentally happen; wherefore (by 5. def. 5.) if H exceed M; K alſo exceeds N; and if equal,

equal; and if leſs, leſs; but if H exceed M; G alſo exceeds L; and if equal, equal; and if leſs, leſs: and G, K are equimultiples of A, E; and L, N any other equimultiples of B, F which may accidentally happen; wherefore (by 5. def. 5.) it is, as A *is* to B ſo *is* E to F.

Wherefore *ratios* which *are* the ſame to the ſame ratio; are the ſame to one another. Which was to be demonſtrated.

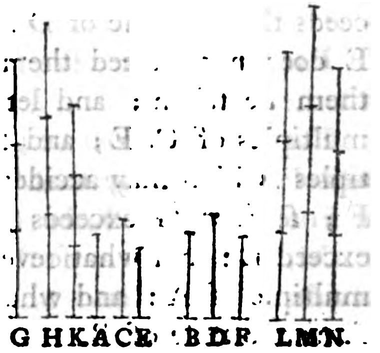
PROP. XII.

If any number of magnitudes be proportionals; it will be, as one of the antecedents *is* to one of the conſequents, ſo *are* all the antecedents to all the conſequents.

Let A, B, C; D, E, F be any number of magnitudes, proportionals; viz. as A *is* to B ſo *let* C *be* to D; and E to F: I ſay that it is as A *is* to B ſo is A, C, E to B, D, F.

For let G, H, K equimultiples of A, C, E be taken; and L, M, N any other which may accidentally happen of B, D, F.

And becauſe it is, as A *is* to B ſo is C to D; and E to F (by ſupp.); and G, H, K have been taken equimultiples of A, C, E; and L, M, N any other equimultiples of B, D, F which may accidentally happen; wherefore (by 5. def. 5.) if G exceed L; H alſo exceeds M and K *exceeds* N; and if equal, equal; and if leſs, leſs: ſo that alſo if G exceed L; G, H, K *taken together* alſo exceeds L, M, N *taken together*; and if *it be* equal; *they are* equal; and if *it be* leſs, *they are* leſs; and G; and G, H, K are equimultiples of A, and of A, C, E; ſince (by 1. 5.) if there be any number of magnitudes equimultiples of an equal number of magnitudes, each of each; whatſoever multiple any one magnitude is of one; the ſame multiple will, all be of all: Certainly for the ſame reaſon alſo; L, and L, M, N are equimultiples of B, and of B, D, F: therefore (by 5. def. 5.) it is, as A *is* to B ſo *is* A, C, E to B, D, F.



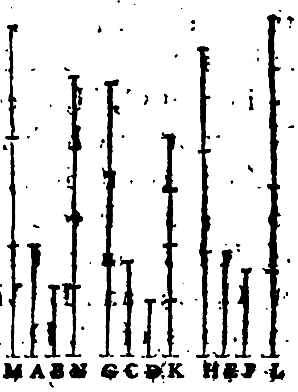
Wherefore if any number of magnitudes be proportionals; it will be as one of the antecedents *is* to one of the conſequents, ſo are all the antecedents to all the conſequents. Which was to be demonſtrated.

PROP. XIII.

If the firſt have the ſame ratio to the ſecond which the third has to the fourth: but if the third have a greater ratio to the fourth than the fifth *has* to the ſixth; the firſt alſo will have a greater ratio to the ſecond than the fifth *has* to the ſixth.

For let the firſt A have the ſame ratio to the ſecond B, *which* the third C has to the fourth D: but let the third C have a greater ratio to the fourth D, than the fifth E, has to the ſixth F; I ſay that the firſt A will have a greater ratio to the ſecond B, than the fifth E has to the ſixth F.

For becauſe C has a greater ratio to D (by ſupp.), than E has to F; (by 7. def. 5.) there are ſome equimultiples of C and E; and other equimultiples, [which may accidentally happen] of D and F: and the multiple of C exceeds the multiple of D: but the multiple of E does not exceed the multiple of F: let them be taken; and let G, H be the equimultiples of C, E; and K, L other equimultiples [which may accidentally happen] of D, F; ſo that G exceeds K, but H does not exceed L: and whatſoever multiple G is of C, let M be the ſame multiple of A: and whatſoever multiple K is of D, let N be the ſame multiple of B.



And becauſe it is, as A *is* to B ſo *is* C to D; and M *and* G have been taken equimultiples of A, C; and N, K other equimultiples [which may accidentally happen] of B, D: wherefore (by 5. def. 5.) if M exceed N, G alſo exceeds K; and if equal, equal: and if leſs, leſs: But G exceeds K (by ſupp.) therefore alſo M exceeds N: but H does not exceed L; and M, H are equimultiples of A, E; and N, L other equimultiples [which may accidentally happen] of B, F; therefore (by 7. def. 5.) A has a greater ratio to B, than E *has* to F.

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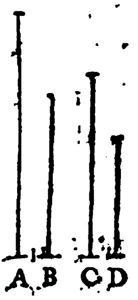
Wherefore if the firſt have the ſame ratio to the ſecond which the third has to the fourth; but *if* the third have a greater ratio to the fourth than the fifth *has* to the ſixth; the firſt alſo will have a greater ratio to the ſecond than the fifth *has* to the ſixth. Which was to be demonſtrated.

PROP. XIV.

If the firſt have the ſame ratio to the ſecond, *which* the third *has* to the fourth; and if the firſt be greater than the third; alſo the ſecond will be greater than the fourth; and if equal, equal: and if leſs, leſs.

For let the firſt A have the ſame ratio to the ſecond B, *which* the third C *has* to the fourth D; and let A be greater than C: I ſay that B is greater than D.



For becauſe (by ſupp.) A is greater than C; and B is any other magnitude which may happen: therefore (by 8. 5.) A has a greater ratio to B, than C *has* to B; but as A *is* to B (by ſupp.) ſo *is* C to D; therefore (by 13. 5.) C has a greater ratio to D, than C *has* to B: but (by 10. 5.) that *magnitude* is the leſs, to which the ſame has a greater ratio; therefore D is leſs than B; ſo that B is greater than D.

Certainly in the ſame manner we ſhall demonſtrate, than when A is equal to C, B will alſo be equal to D: and if A be leſs than C, B will be leſs than D.

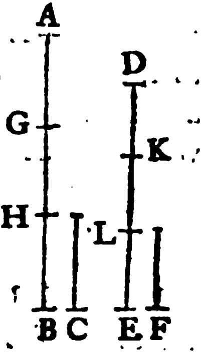
Wherefore if the firſt have the ſame ratio to the ſecond, which the third *has* to the fourth; and if the firſt be greater than the third; alſo the ſecond will be greater than the fourth; and if equal, equal; and if leſs, leſs. Which was to be demonſtrated.

PROP. XV.

The parts, compared with one another, have the ſame ratio *which* their equimultiples *have*.

For let AB be the ſame multiple of C *which* DE *is* of F: I ſay that it is, as C *is* to F ſo *is* AB to DE.

For



For becauſe (by ſupp.) AB is the ſame multiple of C *which* DE *is* of F; therefore as many magnitudes as there are in AB equal to C, ſo many *is there* alſo in DE equal to F; let AB be divided into magnitudes equal to C, viz. AG, GH, HB; and DE into *magnitudes* equal to F, viz. DK, KL, LE: certainly the number of *parts* AG, GH, HB will be equal to the number of *parts* DK, KL, LE: and becauſe AG, GH, HB are equal to one another; alſo DK, KL, LE are equal to one another; therefore it is, (by 7. 5.) as AG *is* to DK ſo *is* GH to KL, and HB to LE; wherefore (by 12. 5.) it will be as one of the antecedents *is* to one of the conſequents, ſo *are* all the antecedents to all the conſequents; wherefore it is, as AG *is* to DK ſo *is* AB to DE; but AG *is* equal to C, and DK *is equal* to F; therefore it is, as C *is* to F ſo *is* AB to DE.

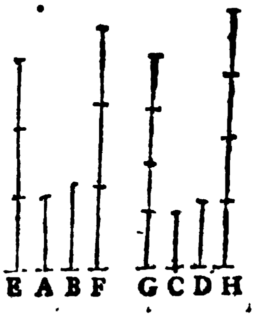
Wherefore the parts, compared with one another, have the ſame ratio *which* their equimultiples *have*. Which was to be demonſtrated.

PROP. XVI.

If four magnitudes be proportionals; they will alſo be alternately proportionals.

Let the four magnitudes, A, B, C, D be proportionals; viz. as A *is* to B ſo *let* C *be* to D: I ſay that they will be alternately proportionals: viz. as A is to C ſo *will* B *be* to D.

For let E, F be taken equimultiples of A, B; and G, H any other equimultiples of C, D which may accidentally happen.



And becauſe E *is* the ſame multiple of A *which* F *is* of B; and (by 15. 5.) parts, compared with one another have the ſame ratio *which* their equimultiples *have*: therefore it is, as A *is* to B ſo *is* E to F; but (by ſupp.) as A *is* to B ſo *is* C to D; and therefore as C *is* to D ſo (by 11. 5.) is E to F: Again, becauſe G *and* H are equimultiples of C *and* D; therefore (by 15. 5.) C *is* to D as G *is* to H; but as C is to D ſo is E to

F; therefore (by 11. 5.) as E is to F ſo is G to H; but if four magnitudes be proportionals, and the firſt be greater than the third; the ſecond (by 14. 5.) will be greater than the fourth; and if equal, equal: and if leſs, leſs: wherefore if E exceed G, F alſo exceeds H: and if equal, equal: and if leſs, leſs: and E and F are equimultiples of A *and* B, and G and H any other equimultiples of C *and* D which may accidentally happen; therefore (by 5. def. 5.) as A *is* to C ſo *is* B to D.

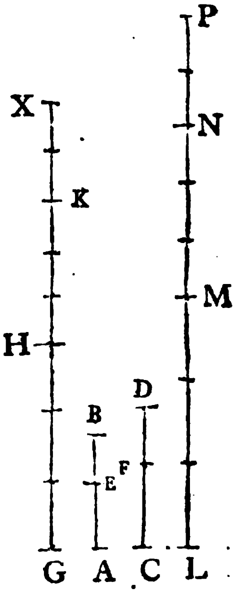
Wherefore if four magnitudes be proportionals they will be (by 13. def. 5.) alternately proportionals. Which was to be demonſtrated.

PROP. XVII.

If compounded magnitudes be proportionals; they will alſo be proportionals *when* divided.

Let the compounded magnitudes, AB, BE, CD, DF be proportionals; viz. as AB is to BE ſo *let* CD *be* to DF: I ſay that they will be proportionals when divided; that is (by 16. def. 5.) as AE *is* to EB ſo *will* CF *be* to FD.

For let GH, HK, LM, MN be taken equimultiples of AE, EB, CF, FD: and *let* KX, NP *be taken* any other equimultiples of EB, FD which may accidentally happen.



And becauſe GH is the ſame muitiple of AE *which* HK *is* of EB; therefore GH is the ſame multiple of AE *which* GK *is* of AB (by 1. 5.); but GH is the ſame multiple of AE *which* LM *is* of CF: therefore GK is the ſame multiple of AB *which* LM *is* of CF. Again becauſe LM is the ſame multiple of CF *which* MN *is* of FD: therefore (by 1. 5.) LM is the ſame multiple of CF *which* LN *is* of CD: but LM was the ſame multiple of CF *which* GK *is* of AB: therefore GK is the ſame multiple of AB *which* LN *is* of CD: therefore GK, LN are equimultiples of AB, CD. Again becauſe HK is the ſame multiple of EB *which* MN *is* of FD; and KX is alſo the ſame multiple of EB *which* NP *is* of FD: alſo (by 2. 5.) HX is the ſame multiple of EB *which*

MP *is* of FD: and becauſe it is (by ſupp.) as AB *is* to BE ſo *is* CD to DF; and GK, LN have been taken equimultiples of AB, CD; and HX, MP [any other] equimultiples of EB, FD [which may accidentally happen]; therefore (by 5. def. 5.) if GK exceed HX, LN alſo exceeds MP: and if equal, equal: and if leſs, leſs: Let GK exceed HX, and, the common *part* HK being taken away, therefore alſo GH exceeds KX: But if GK exceed HX, LN alſo exceeds MP; therefore alſo LN exceeds MP; and MN *which is* common being taken away; LM alſo exceeds NP: ſo that if GH exceed KX; LM alſo exceeds NP. Certainly in the ſame manner we ſhall demonſtrate that if GH be equal to KX; LM will alſo be equal to NP: and if leſs, leſs: But GH, LM are equimultiples of AE, CF; and KX, NP any other equimultiples of EB, FD which may accidentally happen; therefore (by 5. def. 5.) it is, as AE *is* to EB ſo *is* CF to FD.

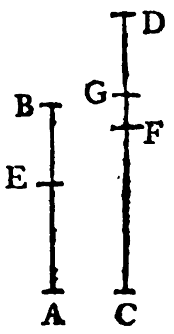
Wherefore if compounded magnitudes be proportionals; they will alſo be proportionals *when* divided. Which was to be demonſtrated.

PROP. XVIII.

If magnitudes *when* divided are proportionals; they will be proportionals *when* compounded.

Let, AE, EB, CF, FD the divided magnitudes be proportionals, viz. as AE *is* to EB ſo *let* CF *be* to FD: I ſay alſo that they will be proportionals *when* compounded, *that is* (by 15. def. 5,) as AB *is* to BE ſo *will* CD *be* to DF.

For if it is not, as AB is to BE ſo *is* CD to DF; it will be as AB *is* to BE ſo *is* CD, either to ſome *magnitude* leſs than FD, or to *one* greater.



Let it be firſt to DG a leſs: and becauſe it is (by this ſupp.) as AB *is* to BE ſo is CD to DG, the compounded magnitudes are proportionals: ſo that alſo (by 17. 5.) they will be proportionals when divided; therefore it is, as AE is to EB ſo *is* CG to GD: but it is alſo ſuppoſed that as AE is to EB ſo is CF to FD: wherefore (by 11. 5.) as CG *is* to GD ſo *is* CF to FD: but CG the firſt *is* greater than CF the third; therefore alſo (by 14. 5.) the ſecond GD is greater than the fourth FD; but it is alſo leſs: which is impoſſible: wherefore it is not, as AB *is* to BE ſo *is* CD to a *magnitude* leſs than DF: Certainly in the ſame manner we ſhall demonſtrate, that neither *is* it to *one* greater. Therefore, it is to *that one* itſelf.

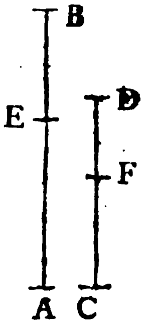
Wherefore if magnitudes *when* divided are proportionals; they will be proportionals *when* compounded. Which was to be demonſtrated.

PROP. XIX.

If it be, as the whole *magnitude is* to the whole, ſo *is* a *magnitude* taken away to a *magnitude* taken away; the remainder alſo will be to the remainder as the whole *is* to the whole.

For let it be, as the whole AB *is* to the whole CD, ſo *let* AE a *magnitude* taken away *be* to CF a *magnitude* taken away; I ſay that EB the remainder will alſo be to FD the remainder, as the whole AB *is* to the whole CD.

For ſince it is, as the whole AB *is* to the whole CD ſo *is* AE to CF: alſo alternately (by 16. 5.) as AB *is* to AE ſo *is* CD to CF; and becauſe the compounded magnitudes are proportionals, they will alſo be proportionals when divided (by 17. 5.); therefore as BE is to EA ſo is DF to FC; and therefore alternately (by 16. 5.), as BE *is* to DF ſo *is* EA to FC: but as EA is to FC ſo is the whole AB ſuppoſed *to be* to the whole CD; therefore alſo (by 11. 5.) the remainder EB will be to the remainder FD as the whole AB is to the whole CD.



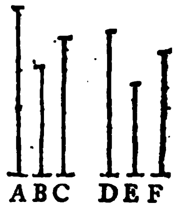
Wherefore if it be, as the whole *magnitude is* to the whole, ſo is a *magnitude* taken away to a *magnitude* taken away; the remainder alſo will be to the remainder as the whole *is* to the whole. Which was to be demonſtrated.

Cor. And becauſe it has been demonſtrated (in this prop.) as AB is to CD ſo *is* EB to FD: and alternately (by 16. 5.) as AB *is* to EB ſo *is* CD to FD: therefore the compounded magnitudes are proportionals: but it has been demonſtrated (in this and prop. 16. 5.) as AB is to AE ſo is CD to CF; and *this* is the converſion *of the proportionals*. Certainly from this it is manifeſt, that if compounded magnitudes be proportionals, they will be proportionals by converſion (by 17. def. 5.). Which was to be demonſtrated.

PROP. XX.

If there be three magnitudes and others, equal to them in multitude, in the ſame ratio, taken two and two; and by equality, if the firſt be greater than the third; alſo the fourth will be greater than the ſixth: and if equal, equal: and if leſs, leſs.

Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude, in the ſame ratio taken two and two; viz. as A is to B ſo *let* D *be* to E, and as B is to C ſo *let* E *be* to F: and by equality, let A be greater than C; I ſay alſo that D will be greater than F: and if equal, equal: and if leſs, leſs.

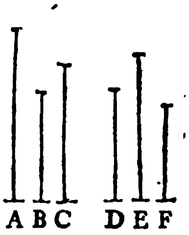


For becauſe A is greater than C, and B *is* any *magnitude* which may accidentally happen: but (by 8. 5.) the greater has a greater ratio to the ſame than the leſs *has*: therefore A has a greater ratio to B than C *has* to B: but (by ſupp.) as A is to B ſo is D to E: but by inverſion (by cor. to 4. 5.) C is to B as F is to E: therefore (by 11. et 13. of 5.) D has a greater ratio to E than F has to E: but of *magnitudes* having ratio to the ſame; that is the greater which has the greater ratio (by 10. 5.): therefore D is greater than F: Certainly in the ſame manner we ſhall demonſtrate, that if A be equal to C, D will alſo be equal to F: and if leſs, leſs.

Wherefore if there be three magnitudes, and others equal to them in multitude, in the ſame ratio, taken two and two; and by equality, if the firſt be greater than the third; alſo the fourth will be greater than the ſixth; and if equal, equal: and if leſs, leſs. Which was to be demonſtrated.

PROP. XXI.

If there be three magnitudes, and others equal to them in multitude; and in the ſame ratio, taken two and two, and their proportion be perturbate; and if by equality the firſt be greater than the third, the fourth will alſo be greater than the ſixth: and if equal, equal: and if leſs, leſs.



Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude, and in the ſame ratio, taken two and two; and let their proportion be perturbate, viz. as A *is* to B ſo *let* E *be* to F; and as B *is* to C ſo *let* D *be* to E: and by equality let A be greater than C: I ſay alſo that D will be greater than F: and if equal, equal: and if leſs, leſs.

For becauſe A is greater than C, and B is ſome other *magnitude*: therefore (by 8. 5.) A has a greater ratio to B than C *has* to B: but as A *is* to B ſo (by ſupp.) is E to F: and as C is to B ſo by inverſion *is* E to D: therefore (by 11. and 13. 5.) E has a greater ratio to F than E has to D: but (by 10. 5.) that is the leſs *magnitude* to which the ſame has a greater ratio; therefore F is leſs than D: wherefore D is greater than F: Certainly we ſhall demonſtrate in the ſame manner, that if equal, equal; certainly if A be equal to C, D alſo will be equal to F: and if leſs, leſs.

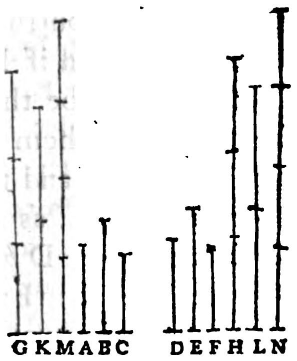
Wherefore if there be three magnitudes and others equal to them in multitude; and in the ſame ratio, taken two and two, and their proportion be perturbate; and *if* by equality the firſt be greater than the third, the fourth will alſo be greater than the ſixth: and if equal, equal: and if leſs, leſs. Which was to be demonſtrated.

PROP. XXII.

If there be any number of magnitudes; and others equal to them in multitude; in the ſame ratio, taken two and two; alſo by equality they will be in the ſame ratio.

Let there be any number of magnitudes A, B, C; and D, E, F others equal to them in multitude; in the ſame ratio, taken two and two; viz. as A *is* to B ſo *let* D *be* to E and as B *is* to C ſo let E *be* to F: I ſay alſo, that by equality they will be in the ſame ratio; viz. (by 18. def. 5.) as A *is* to C ſo *will* D *be* to F.

For let G, H be taken equimultiples of A, D; and K, L any other equimultiples of B, E which may accidentally happen; and beſides, M, N any other equimultiples of C, F which may accidentally happen.



And becauſe it is (by ſupp.) as A *is* to B ſo *is* D to E; and G, H have been taken equimultiples of A, D; and K, L any other equimultiples of B, E which may accidentally happen: therefore (by 4. 5.) it is, as G *is* to K ſo *is* H to L. Certainly for the ſame reaſon alſo as K *is* to M ſo *is* L to N: wherefore becauſe there are three magnitudes G, K, M; and others equal to them in multitude H, L, N and in the ſame ratio, taken two and two; therefore by equality, (by 20. 5.) if G exceed M, alſo H exceeds N: and if equal, equal: and if leſs, leſs: and G and H are equimultiples of A, D; and M, N any other equimultiples of C, F which may accidentally happen: therefore (by 5. def. 5.) it is, as A *is* to C ſo *is* D to F.

Wherefore if there be any number of magnitudes, and others equal to them in multitude; in the ſame ratio, taken two and two; alſo by equality, they will be in the ſame ratio. Which was to be demonſtrated.

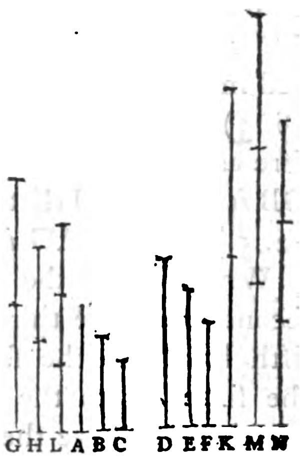
PROP. XXIII.

If there be three magnitudes, and others equal to them in multitude; in the ſame ratio, taken two and two; and *if* their proportion be perturbate; they will alſo be in the ſame ratio by equality.

Let A, B, C be three magnitudes, and D, E, F others equal to them in multitude; in the ſame ratio, taken two and two; and let their proportion be perturbate, viz. as A *is* to B ſo *let* E *be* to F; and as B *is* to C ſo *let* D *be* to E: I ſay that it is, as A *is* to C ſo *is* D to F.

For let G, H, K be taken equimultiples of A, B, D; and L, M, N any other equimultiples of C, E, F which may accidentally happen.

And becauſe G, K are equimultiples of A, B; and parts, (by 15. 5.) have the ſame ratio *to one another* as their equimultiples; therefore it is, as A *is* to B ſo *is* G to H: Certainly for the ſame reaſon alſo as E *is* to F ſo *is* M to N: and becauſe it is, as B *is* to C ſo



is D to E; and H, K have been taken equimultiples of B, D; and L, M any other equimultiples of C, E which may accidentally happen; therefore it is, (by 4. 5.) as H *is* to L ſo *is* K to M: but it has been demonſtrated as G *is* to H ſo *is* M to N: wherefore becauſe there are three magnitudes G, H, L and others K, M, N equal to them in multitude; in the ſame ratio taken two and two; and their proportion is perturbate; therefore by equality (by 21. 5.) if G exceed L; K alſo exceeds N: and if equal, equal: and if leſs, leſs: and G, K are equimultiples of A, D; and L, N *any other* equimultiples of C, F *which may accidentally happen*; therefore (by 5. def. 5.) A *is* to C as D *is* to F.

Wherefore if there be three magnitudes, and others equal to them in multitude; in the ſame ratio taken two and two; and if their proportion be perturbate; they will alſo be in the ſame ratio by equality. Which was to be demonſtrated.

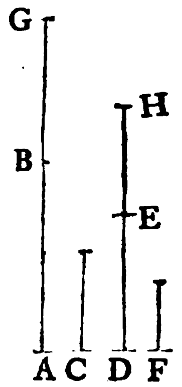
PROP. XXIV.

If the firſt have the ſame ratio to the ſecond *which* the third *has* to the fourth; and if the fifth have alſo the ſame ratio to the ſecond *which* the ſixth *has* to the fourth; alſo the firſt and fifth together will have the ſame ratio to the ſecond *which* the third and ſixth *together has* to the fourth.

For let the firſt AB have the ſame ratio to the ſecond C, *which* the third DE *has* to the fourth F: and alſo let the fifth BG have the ſame ratio to the ſecond C, *which* the ſixth EH *has* to the fourth F: I ſay that alſo AG the firſt together with the fifth will have the ſame ratio to C the ſecond *which* DH the third *together* with the ſixth *has* to F the fourth.

For becauſe it is, as BG *is* to C ſo is EH to F: therefore by inverſion (by cor. to 4. 5.) C *is* to BG as F *is* to EH, Wherefore becauſe it is as AB *is* to C ſo *is* DE to F; and as C *is* to BG ſo *is*

F to EH: therefore by equality (by 22. 5.) it is, as AB *is* to BG ſo *is* DE to EH: and becauſe the divided magnitudes are proportionals, they will alſo (by 18. 5) be proportionals *when* compounded: wherefore as AG is to GB ſo is DH to HE: but it is, as GB *is* to C ſo is HE to F: therefore by equality (by 22. 5.) it is, as AG *is* to C ſo *is* DH to F.

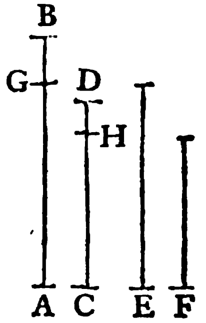


Wherefore if the firſt have the ſame ratio to the ſecond *which* the third *has* to the fourth; and *if* the fifth have alſo the ſame ratio to the ſecond *which* the ſixth *has* to the fourth; alſo the firſt and fifth together will have the ſame ratio to the ſecond *which* the third and ſixth together has to the fourth. Which was to be demonſtrated.

PROP. XXV.

If four magnitudes be proportionals; the greateſt and the leaſt of them are greater than the other two.

Let the four magnitudes AB, CD, E, F be proportionals; viz. as AB *is* to CD ſo *let* E *be* to F: and let AB be the greateſt of them and F the leaſt; I ſay that AB *and* F are greater than CD *and* E.



For let AG be made equal to E and *let* CH *be made* equal to F.

Wherefore becauſe it is, as AB *is* to CD ſo *is* E to F; and AG is equal to E and CH to F; therefore it is as AB is to CD ſo is AG to CH: and becauſe it is as the whole AB *is* to the whole CD ſo is AG taken away to CH taken away; therefore (by 19. 5.) the remainder GB will be to the remainder HD as the whole AB is to the whole CD: but (by ſupp.) AB is greater than CD; therefore GB is greater than HD: and becauſe AG is equal to E and CH to F: therefore AG and F *together* are equal to CH and E *together*; and becauſe if equals be added to unequals, the wholes are unequal: wherefore GB and HD being unequal: and AG and F be added to the greater GB: and CH and E be added to the *leſs* HD: AB and F together are greater than CD and E.

Wherefore if four magnitudes be proportionals, the greateſt and the leaſt of them are greater than the other two. Which was to be demonſtrated.

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THE

ELEMENTS

OF

EUCLID.

BOOK VI.

DEFINITIONS.

1. SIMILAR rectilineal figures are thoſe which have their angles equal each to each; and the ſides about the equal angles proportionals. 2. But thoſe are reciprocal figures; when the antecedent and conſequent terms are in each of the figures.

3. A ſtraight line is ſaid to have been cut in extreme and mean ratio, when it is, as the whole *line* is to the greater ſegment, ſo is the greater *ſegment* to the leſs.

4. The altitude of any figure is the perpendicular drawn from the vertex to the baſe.

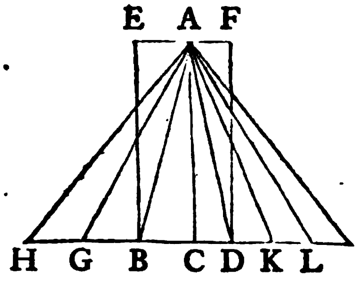
5. A ratio is ſaid to be compounded of ratios, when the quantities of the ratios being multiplied into one another do make ſome *ratio*.

PROP. I.

The triangles and parallelograms which are under the ſame altitude are to one another as *their* baſes.

Let ABC, ACD be triangles; and EC, CF parallelograms, which are under the ſame altitude, viz. the perpendicular drawn from *the point* A to BD: I ſay that it is, as the baſe BC *is* to the baſe CD ſo *is* the triangle ABC to the triangle ACD: alſo ſo *is* the parallelogram EC to the parallelogram CF.

For let BD be produced towards both parts to the points H, L: and let any number of *lines* BG, GH be made equal to the baſe BC: and any number of *lines* DK, KL equal to the baſe CD: and let AG, AH, AK, AL be joined.



And becauſe CB, BG, GH are equal to one another; alſo (by 38. 1.) the triangles AGH, AGB, ABC are equal to one another: therefore whatſoever multiple the baſe HC is of the baſe BC; the ſame multiple alſo is the triangle AHC of the triangle ABC. Certainly for the ſame reaſon alſo, whatſoever multiple the baſe CL is of the baſe CD the ſame multiple is the triangle ACL of the triangle ACD. And if the baſe HC be equal to the baſe CL (by 38. 1.) the triangle AHC is alſo equal to the triangle ACL: and if the baſe HC exceed the baſe CL; the triangle AHC alſo exceeds the triangle ACL: and if leſs, leſs. There being four magnitudes, the two baſes BC, CD; and the two triangles ABC, ACD: *and* equimultiples of the baſe BC and of the triangle ABC have been taken viz. the baſe HC and the triangle AHC: and of the baſe CD and of the triangle ACD any other equimultiples which may accidentally happen viz. the baſe CL and the triangle ACL: and it has been demonſtrated that if the baſe HC exceed the base CL; the triangle AHC alſo exceeds the triangle ACL: and if equal, equal: and if leſs, leſs: therefore (by 5. def. 5.) it is, as, the baſe BC *is* to the baſe CD ſo *is* the triangle ABC to the triangle ACD.

And becauſe the parallelogram EC is double of the triangle ABC (by 41. 1.): and the parallelogram FC is double of the triangle ACD; and parts (by 15. 5.) have the ſame ratio *to one another* as their equimultiples; therefore it is, as the triangle ABC is to the triangle ACD ſo *is* the parallelogram EC to the parallelogram FC: wherefore becauſe it has been demonſtrated *that* as the baſe BC *is* to the baſe CD ſo is the triangle ABC to the triangle ACD: and as the triangle ABC *is* to the triangle ACD ſo is the parallelogram EC to the parallelogram FC; therefore alſo (by 11. 5.) the baſe BC is to the baſe CD as the parallelogram EC *is* to the parallelogram FC.

Wherefore the triangles and parallelograms, which are under the ſame altitude, are to one another as *their* baſes. Which was to be demonſtrated.

PROP. II.

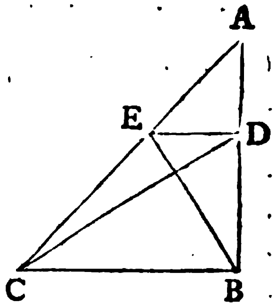
If any ſtraight line be drawn parallel to one of the ſides of a triangle, it will cut the ſides of the triangle proportionally: and if the ſides of the triangle be cut proportionally, the ſtraight line joining the ſections will be parallel to the remaining ſide of the triangle.

For let DE be drawn parallel to BC one of the ſides of the triangle ABC: I ſay that it is, as BD *is* to DA ſo *is* CE to EA.

For let BE, CD be joined.

Therefore the triangle BDE is equal to the triangle CDE; for they are upon the ſame baſe DE and between the ſame parallel lines DE, BC: but ADE is ſome other triangle; and (by 7. 5.) equal *magnitudes* have the ſame ratio to the ſame *magnitude*: therefore it is, as the triangle BDE *is* to the triangle ADE ſo *is* the triangle CDE to the triangle ADE: but as the triangle BDE is to the triangle ADE ſo *is* (by 1. 6.) BD to DA: for being under the ſame altitude, the perpendicular drawn from the *point* E to AB, they are to one another as their baſes. Certainly for the ſame reaſon alſo, as the triangle CDE *is* to the triangle ADE ſo *is* CE to EA: Therefore (by 11. 5.) as BD is to DA ſo is CE to EA.

But let AB, AC, the ſides of the triangle ABC be cut proportionally in the points D, E viz. as BD *is* to DA ſo *let* CE *be* to EA:



and let DE be joined: I ſay that DE is parallel to BC.

For the ſame things being conſtructed; becauſe it is, as BD *is* to DA ſo *is* CE to EA: but as BD *is* to DA ſo (by 1. 6.) *is* the triangle BDE to the triangle ADE: and as CE is to EA ſo is the triangle CDE to the triangle ADE: therefore alſo (by 11. 5.) as the triangle BDE *is* to the triangle ADE ſo *is* the triangle CDE to the triangle ADE: therefore each of the triangles BDE, CDE has the ſame ratio to the triangle ADE; therefore (by 9. 5.) the triangle BDE is equal to the triangle CDE: and they are upon the ſame baſe DE: but equal triangles being upon the ſame baſe are alſo (by 39. 1.) between the ſame parallels: therefore DE is parallel to BC.

Wherefore if any ſtraight line be drawn parallel to one of the ſides of a triangle, it will cut the ſides of the triangle proportionally: and if the ſides of the triangle be cut proportionally, the ſtraight line joining the ſections will be parallel to the remaining ſide of the triangle. Which was to be demonſtrated.

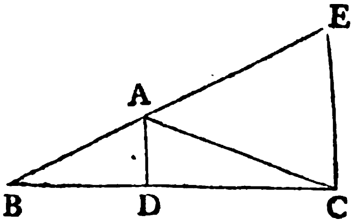
PROP. III.

If an angle of a triangle be cut in halves, and the ſtraight line cutting the angle alſo cut the baſe; the ſegments of the baſe will have the ſame ratio *which* the remaining ſides of the triangle *have to one another*: and if the ſegments of the baſe have the ſame ratio *which* the remaining ſides of the triangle *have to one another*; the ſtraight line joining the vertex *and the point of* ſection cuts the angle of the triangle in halves.

Let ABC be a triangle, and let the angle BAC be cut in halves by the ſtraight line AD: I ſay that it is as BD *is* to DC ſo *is* BA to AC.

For let CE be drawn through the point C parallel to DA: and BA being produced let it meet *CE* in E.

And becauſe the ſtraight line AC hath fallen upon the parallels AD, EC; therefore the angle ACE (by 29. 1.) is equal to the angle



angle CAD: but the angle CAD is ſuppoſed equal to BAD: therefore the angle BAD is equal to the angle ACE. Again becauſe the ſtraight line BAE hath fallen upon the parallels AD, EC; the outward angle BAD is equal (by 29. 1.) to the inward *angle* AEC: but the angle ACE hath been alſo demonſtrated *to be* equal to BAD; therefore (by com. not. 1.) the *angle* ACE is equal to AEC: ſo that alſo (by 6. 1.) the ſide AE is equal to the ſide AC: and becauſe AD has been drawn *parallel* to CE one of the ſides of the triangle BCE; there is *this* proportion therefore (by 2. 6.) as BD is to DC ſo *is* BA to AE: But AE is equal to AC: therefore it is (by 7. 5.) as BD *is* to DC ſo *is* BA to AC.

But let BD be to DC as BA to AC; and let AD be joined: I ſay that the angle BAC hath been cut in halves by the ſtraight line AD.

For the ſame things being conſtructed; becauſe it is (by ſupp.), as BD *is* to DC ſo *is* BA to AC: but alſo (by 2. 6.) as BD *is* to DC ſo *is* BA to AE; for AD hath been drawn parallel to CE one of the ſides of the triangle BCE: and therefore (by 11. 5.) as BA *is* to AC ſo *is* BA to AE; therefore (by 9. 5.) AC is equal to AE; ſo that alſo (by 5. 1.) the angle AEC is equal to the angle ACE: but the *angle* AEC is equal to the outward *angle* BAD (by 29. 1.); and alſo the *angle* ACE is equal to the alternate *angle* CAD; therefore the *angle* BAD is equal to CAD: wherefore the *angle* BAC hath been cut in halves by the ſtraight line AD.

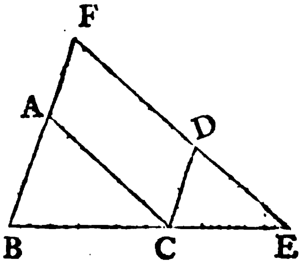
Wherefore if an angle of a triangle be cut in halves; and the ſtraight line cutting the angle alſo cut the baſe; the ſegments of the baſe will have the ſame ratio *which* the remaining ſides of the triangle *have to one another*: and if the ſegments of the baſe have the ſame ratio *which* the remaining ſides of the triangle *have to one another*; the ſtraight line, joining the vertex and the point of ſection, cuts the angle of the triangle in halves. Which was to be demonſtrated.

PROP. IV.

The ſides about the equal angles of equiangular triangles are proportionals; and thoſe ſides are of like ratio *which are* extended under the equal angles.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE; and the angle ACB *equal* to the angle DEC; and beſides the angle BAC equal to the angle CDE: I ſay that the ſides about the equal angles of the triangles ABC, DCE are proportionals; and *that* thoſe ſides are of like ratio *which are* extended under the equal angles.

For let BC be placed in a ſtraight line with CE: and becauſe the angles ABC, ACB are leſs than two right angles; and ACB is equal to the *angle* DEC; therefore the *angles* ABC, DEC are leſs than two right angles; therefore (by com. not. 11.) BA, ED produced will meet; let them be produced and meet in F.



And becauſe the angle DCE is equal to the *angle* ABC; therefore (by 28. 1.) BF is parallel to CD: Again, becauſe the *angle* ACB is equal to DEC, AC is parallel to FE; therefore FACD is a parallelogram; therefore (by 34. 1.) FA is equal to CD; and AC to FD: and becauſe AC hath been drawn parallel to FE one of the ſides of the triangle FBE; therefore it is, (by 2. 6.) as BA *is* to AF ſo *is* BC to CE: but AF *is* equal to CD; therefore as BA is CD (by 7. 5.) ſo *is* BC to CE; and alternately (by 16. 5.) as AB *is* to BC ſo *is* DC to CE: Again, becauſe CD is parallel to BF; therefore (by 2. 6.) it is, as BC *is* to CE ſo *is* FD to DE: but FD is equal to AC: therefore as BC *is* to CE ſo is AC to DE; therefore alternately, as BC is to CA ſo *is* CE to ED: wherefore becauſe it has been demonſtrated *that* as AB *is* to BC ſo is DC to CE; and as BC *is* to CA ſo *is* CE to ED; therefore by equality (by 22. 5.) *it is*, as BA *is* to AC ſo *is* CD to DE.

Wherefore the ſides about the equal angles of equiangular triangles are proportionals; and the ſides of like ratio are extended under the equal angles. Which was to be demonſtrated.

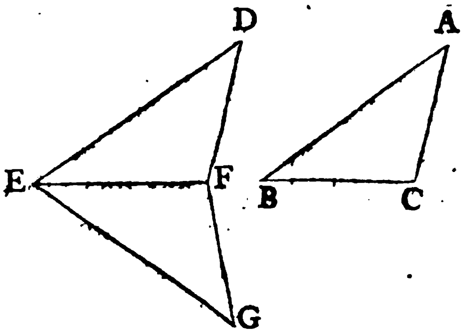
PROP.

PROP. V.

If two triangles have their ſides proportionals, the triangles will be equiangular; and will have thoſe angles equal under which the ſides of like ratio are extended.

Let ABC, DEF be two triangles having their ſides proportionals, viz. as AB *is* to BC ſo *let* DE *be* to EF; and as BC *is* to CA ſo *let* EF *be* to FD; and beſides as BA is to AC ſo *let* ED *be* to DF: I ſay, that the triangle ABC is equiangular to the triangle DEF, and will have thoſe angles equal, under which the ſides of like ratio are extended viz. ABC equal to DEF; and BCA equal to EFD; and beſides BAC equal to EDF.

For with the ſtraight line EF and at the points E, F in it, let the angle FEG be made equal to ABC; and *let* EFG be made equal to BCA; therefore (by 32. 1.) the remaining angle BAC is equal to the remaining angle EGF.



Wherefore the triangle ABC is equiangular to the triangle EGF; therefore the ſides about the equal angles of the triangles ABC, EGF are proportionals (by 4. 6.); and the ſides of like ratio are extended under the equal angles; therefore it is, as AB is to BC ſo is GE to EF: but as AB *is* to BC ſo is DE ſuppoſed *to be* to EF; wherefore (by 11. 5.) as DE is to EF ſo is GE to EF: therefore each of the *lines* DE, GE has the ſame ratio to EF; therefore (by 9. 5.) DE is equal to GE. Certainly for the ſame reaſon alſo DF is equal to GF: wherefore becauſe DE is equal to EG and EF common, certainly the two DE, EF are equal to the two GE, EF; and the baſe DF is equal to the baſe GF; therefore (by 8. 1.) the angle DEF is equal to the angle GEF; and the triangle DEF is equal to the triangle GEF; and the remaining angles *are* equal to the remaining angles, under which the equal ſides are extended (by 4. 1.): therefore the angle DFE is equal to the angle GFE; and EDF to EGF:

EGF : and becauſe DEF is equal to GEF : but GEF (by conſt.) is equal to ABC; therefore alſo the angle ABC is equal to DEF : Certainly for the ſame reaſon alſo ACB is equal to DFE; and beſides (by 32. 1.) the angle at A is equal to the angle at D : Therefore the triangle ABC is equiangular to the triangle DEF.

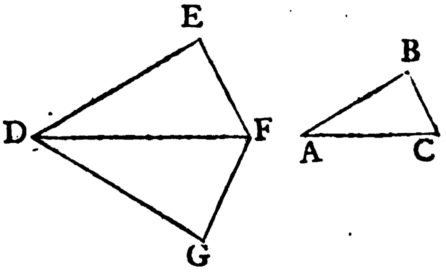
Wherefore if two triangles have their ſides proportionals, the triangles will be equiangular : and will have the angles equal, under which the ſides of like ratio are extended. Which was to be demonſtrated.

PROP. VI.

If two triangles have one angle equal to one angle, and the ſides about the equal angles proportionals : the triangles will be equiangular, and will have the angles equal, under which the ſides of like ratio are extended.

Let ABC, DEF be two triangles, having one angle BAC equal to one angle EDF; and the ſides about the equal angles proportionals viz. as BA is to AC ſo *let* ED *be* to DF : I ſay that the triangle ABC is equiangular to the triangle DEF; and will have the angle ABC equal to the *angle* DEF; and ACB *equal* to DFE.

For with the ſtraight line DF and at the points, D, F in it; let the angle FDG be made equal to either of the angles BAC, EDF; and *let* DFG *be made equal* to ACB.



Therefore the remaining *angle* at B is equal (by 32. 1.) to the remaining *angle* at G : therefore the triangle ABC is equiangular to the triangle DGF; therefore there is this proportion (by 4. 6.) as BA *is* to AC ſo *is* GD to DF; but it is alſo ſuppoſed *that* as BA *is* to AC ſo *is* ED to DF; therefore alſo (by 11. 5.) as ED *is* to DF ſo *is* GD to DF; therefore (by 9. 5.) ED is equal to DG; and DF is common : certainly the two ED, DF are equal to the two GD, DF; and (by conſt.) the angle EDF is equal to the angle GDF : therefore (by 4. 1.) the baſe EF is equal to the baſe FG; and the triangle DEF is equal to the triangle

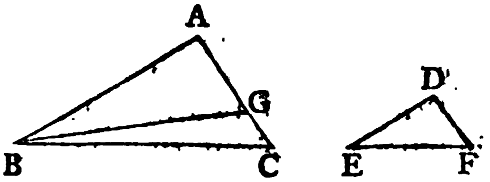
angle DGF; and the remaining angles will be equal to the remaining angles, each to each, under which the equal ſides are extended; therefore the angle DFG is equal to the angle DFE; and the *angle* at G to the *angle* at E; but the angle DFG is equal (by conſt.) to the angle ACB; therefore alſo ACB is equal to DFE: but BAC is alſo ſuppoſed equal to EDF; therefore the remaining *angle* at B is equal to the remaining *angle* at E: wherefore the triangle ABC is equiangular to the triangle DEF.

Wherefore if two triangles have one angle equal to one angle, and the ſides about the equal angles proportionals: the triangles will be equiangular, and will have the angles equal, under which the ſides of like ratio are extended. Which was to be demonſtrated.

PROP. VII.

If two triangles have one angle equal to one angle; and the ſides about *two* other angles proportionals; and if each of the remaining angles *be* at the ſame time leſs or not leſs than a right angle: the triangles will be equiangular; and will have thoſe angles equal, about which the ſides are, *which are* proportionals.

Let ABC, DEF be two triangles having one angle equal to one angle viz. the *angle* BAC *equal* to the *angle* EDF; and the ſides about *two* other angles proportionals, the *angles* ABC, DEF; ſo *that* AB *may be* to BC as DE to EF; and firſt let each of the remaining angles at C and F be at the ſame time leſs than a right angle: I ſay, that the triangle ABC is equiangular to the triangle DEF; and *that* the angle ABC will be equal to the *angle* DEF; and the remaining *angle* viz. the *angle* at C equal to the remaining *angle* at F.



For if the angle ABC be unequal to the angle DEF, one of them is greater: let ABC be the greater: and with the ſtraight line AB, and at the point B in it, let the *angle* ABG be made equal to the angle DEF.

And becauſe the angle A is equal to D; and the angle ABG to DEF; therefore the remaining angle AGB is equal to the remaining angle DFE: therefore the triangle ABG is equiangular to the triangle DEF; therefore it is (by 4. 6.) as AB *is* to BG ſo *is* DE to EF; but as DE is to EF ſo is AB ſuppoſed *to be* to BC; and therefore (by 11. 5.) as AB *is* to BC ſo *is* AB to BG: wherefore AB has the ſame ratio to each of the *lines* BC, BG; therefore (by 9. 5.) BC is equal to BG; ſo that alſo (by 5. 1.) the angle BGC is equal to the angle BCG: but the angle at C is ſuppoſed *to be* leſs than a right angle; therefore BGC is alſo leſs than a right angle; ſo that the angle adjacent to it viz. AGB is (by 13. 1.) greater than a right angle; and it has been demonſtrated to be equal to the angle at F; therefore the angle at F is greater than a right angle; but it is ſuppoſed to be leſs than a right angle; which is abſurd: wherefore the angle ABC is not unequal to the *angle* DEF, therefore *it is* equal: and alſo (by ſupp.) the angle at A is equal to the angle at D; therefore (by 32. 1.) the remaining *angle* at C is equal to the remaining angle at F: Therefore the triangle ABC is equiangular to the triangle DEF.

But again, let each of the angles at C and F be ſuppoſed not *to be* leſs than a right angle: I ſay again that alſo thus the triangle ABC is equiangular to the triangle DEF.

For the ſame things being conſtructed, we ſhall demonſtrate in like manner that BC is equal to BG; ſo that alſo (by 5. 1.) the angle at C is equal to the *angle* BGC: but the angle at C is not leſs than a right angle; therefore neither is the angle BGC leſs than a right angle; therefore the two angles of the triangle BGC are not leſs than two right angles, which (by 17. 1.) is impoſſible: Therefore again the angle ABC is not unequal to the *angle* DEF; therefore *it is* equal: but alſo (by ſupp.) the angle at A is equal to the angle at D; therefore (by 32. 1.) the remaining *angle* at C is equal to the remaining *angle* at F: wherefore the triangle ABC is equiangular to the triangle DEF.

Wherefore if two triangles have one angle equal to one angle; but the ſides about *two* other angles proportionals; and each of the remaining angles, at the ſame time, either leſs or not leſs than a right angle: the triangles will be equiangular and will have the angles equal about which the ſides are, *which are* proportionals. Which was to be demonſtrated.

PROP.

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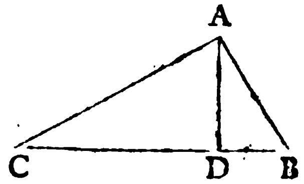
PROP. VIII.

If in a right angled triangle, a perpendicular be drawn from the right angle to the baſe; the triangles at the perpendicular are ſimilar to the whole *triangle* and to one another.

Let ABC be a right angled triangle, having the angle BAC a right angle; and let AD be drawn from the *point* A perpendicular to BC: I ſay that each of the triangles ABD, ADC is ſimilar to the whole ABC, and alſo to one another.



For ſince the angle BAC is equal to the *angle* ADB, for each of them *is* a right angle: and the angle at B is common to the two triangles, both ABC and ABD; therefore (by 32. 1.) the remaining *angle* ACB is equal to the remaining *angle* BAD; therefore the triangle ABC is equiangular to the triangle ABD; wherefore it is, (by 4. 6.) as BC ſubtending the right angle of the triangle ABC *is* to BA ſubtending the right angle of the triangle ABD, ſo *is* the ſame AB ſubtending the angle at C of the triangle ABC to BD ſubtending the angle equal to that at C, viz. the *angle* BAD of the triangle ABD: and moreover ſo *is* AC to AD ſubtending the angle at B common to the two triangles: wherefore the triangle ABC is equiangular to the triangle ABD; and has the ſides about the equal angles proportionals: therefore (by 1. def. 6.) the triangle ABC is ſimilar to the triangle ABD. Certainly in the ſame manner we ſhall demonſtrate, that the triangle ADC is alſo ſimilar to the triangle ABC; therefore each of the triangles ABD, ADC is ſimilar to the whole triangle ABC.

I ſay alſo that the triangles ABD, ADC are ſimilar to one another.

For becauſe the right angle BDA is equal to the right angle ADC: but the *angle* BAD has been demonſtrated *to be* equal to the *angle* at C; therefore alſo the remaining angle at B is equal to the remaining angle DAC: wherefore the triangle ABD is equiangular to the triangle ADC: therefore it is, as BD ſubtending the angle BAD of the triangle ABD is to DA ſubtending the angle at C, of the triangle ADC, equal to the angle BAD ſo is the ſame AD ſubtending the angle at B of the triangle ABD, to DC ſubtending the angle DAC of the triangle ADC, equal to the *angle* at B: and beſides ſo is BA ſubtending the right angle ADB to AC ſubtending the right angle ADC: therefore (by 1. def. 6.) the triangle ABD is ſimilar to the triangle ADC.

Wherefore if in a right angled triangle, a perpendicular be drawn from the right angle to the baſe; the triangles at the perpendicular are ſimilar to the whole *triangle* and to one another. Which was to be demonſtrated.

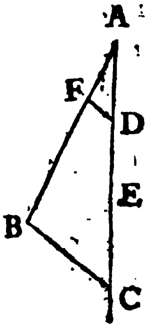
Cor. Certainly it is manifeſt from this, that if a perpendicular be drawn from the right angle in a right angled triangle to the baſe; the *perpendicular* drawn is a mean proportional between the ſegments of the baſe: and beſides the ſide adjacent to the ſegment is a mean proportional between the baſe and any one of the ſegments.

PROP. IX.

To cut off any part required from a given ſtraight line.

Let AB be the given ſtraight line: it is required to cut off any part from AB.

Let a third *part* be required; and let any ſtraight line AC be drawn from the *point* A, containing any angle, which may accidentally happen, with the line AB; and let D be taken in AC any point which may accidentally happen; and (by 3. 1.) let DE, EC be made equal to AD; and let BC be joined; and through the point D, let DF be drawn (by 31. 1.) parallel to BC.



Wherefore becauſe DF has been drawn *parallel* to BC one of the ſides of the triangle ABC; therefore (by 2. 6.) there is this proportion, as CD *is* to DA ſo *is* BF to FA: but CD is the double of DA; therefore BF *is* the double of FA; therefore BA *is* the triple of AF.

Therefore the required third part AF hath been cut off from the given ſtraight line AB. Which was to be done.

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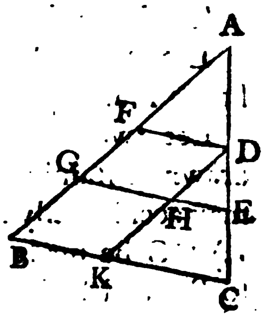
PROP. X.

To cut an undivided ſtraight line, in like manner as a given ſtraight line hath been cut.

Let AB be the given undivided ſtraight line, and AC the one which hath been cut: it is required to cut the undivided ſtraight line AB in like manner as AC has been cut.

Let AC be cut in the points D, E: and let *the lines* be placed ſo as to contain any angle which may accidentally happen; and let BC be joined; and let DF, GE be drawn (by 31. 1.) through the *points* D, E parallel to BC: and through the *point* D, let DHK be drawn parallel to AB.



Therefore each of the *figures* FH, HB is a parallelogram: therefore (by 34. 1.) DH *is* equal to FG; and HK to GB: And becauſe HE hath been drawn *parallel* to KC one of the ſides of the triangle DKC; (by 2. 6.) there is *this* proportion, as CE *is* to ED ſo *is* KH to HD; but KH is equal to BG, and HD to GF; therefore it is (by 7. 5.) as CE is to ED ſo *is* BG to GF: Again, becauſe FD hath been drawn parallel to EG one of the ſides of the triangle AGE; therefore (by 2. 6.) there is *this proportion*, as ED *is* to DA ſo *is* GF to FA; but it has been alſo demonſtrated *that* as CE *is* to ED ſo is BG to GF: It is therefore as CE *is* to ED ſo *is* BG to GF; and as ED *is* to DA ſo *is* GF to FA.

Wherefore the given undivided ſtraight line AB hath been cut in like manner as the given ſtraight line AC had been cut. Which was to be done.

PROP. XI.

To find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines; and let them be placed containing any angle which may accidentally happen: it is required to find a third proportional to AB, AC.

For let AB, AC be produced to the points D, E; and let BD be made equal to AC; and let BC be joined; and through the point

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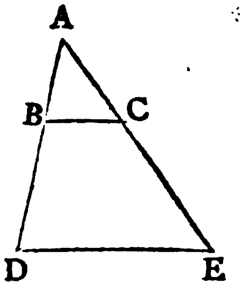
D, let DE be drawn parallel to it.



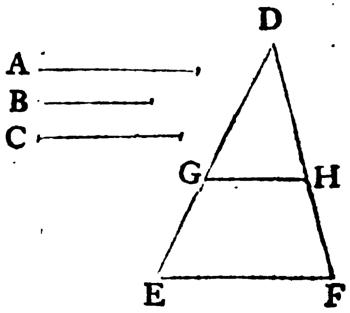
Wherefore becauſe BC hath been drawn *parallel* to DE one of the ſides of the triangle ADE, there is *this* proportion (by 2. 6.), as AB is to BD ſo *is* AC to CE : but BD is equal to AC (by conſt.); therefore it is, as AB *is* to AC ſo is AC to CE.

Wherefore two ſtraight lines AB, AC being given, CE a third proportional to them hath been found. Which was to be done.

P R O P. XII.

To find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines ; it is required to find a fourth proportional to the ſtraight lines A, B, C.



Let two ſtraight lines DE, DF be drawn containing any angle EDF which may accidentally happen: and let (by 3. 1.) DG be made equal to A, GE equal to B; and beſides DH equal to C: and GH being joined, let EF be drawn through E parallel to it.

Therefore becauſe GH hath been drawn parallel to EF one of the ſides of the triangle DEF; therefore it is, (by 2. 6.) as DG *is* to GE ſo *is* DH to HF; but (by the conſt.) DG is equal to A, GE to B, and and DH to C; therefore it is, as A *is* to B ſo *is* C to HF.

Wherefore three ſtraight lines A, B, C being given, HF a fourth proportional has been found. Which was to be done.

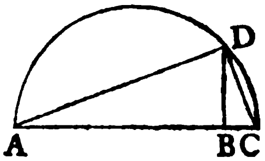
P R O P. XIII.

To find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines : it is required to find a mean proportional between AB, BC.

Let them be placed in a ſtraight line; and let the ſemicircle ADC be deſcribed upon AC; and let BD be drawn from the point

B at right angles to the ſtraight line AC; and let AD, DC be joined.



And becauſe the angle ADC is in a ſemicircle (by 31. 3.) it is a right angle: and becauſe in a right angled triangle ADC, DB hath been drawn from the right angle perpendicular to the baſe therefore (by cor. to 8. 6.) DB is a mean proportional between AB, BC the ſegments of the baſe.

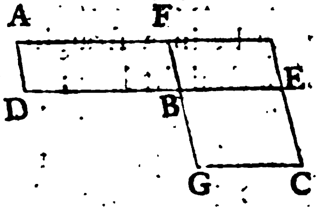
Wherefore two ſtraight lines AB, BC being given, DB a mean proportional *between them* hath been found. Which was to be done.

PROP. XIV.

The ſides about the equal angles of equal parallelograms which have one angle equal to one angle are reciprocally proportional: and thoſe parallelograms, having one angle equal to one angle, of which the ſides about the equal angles are reciprocally proportional, are equal,

Let AB, BC be equal parallelograms, having the angles at B equal; and let DB, BE be placed in a ſtraight line; therefore (by 14. 1.) FB, BG are in a ſtraight line: I ſay that the ſides of the *parallelograms* AB, BC, which are about the equal angles, are reciprocally proportional, that is, as DB is to BE ſo is GB to BF.

For let the parallelogram FE be compleated.



And becauſe the parallelogram AB is equal (by ſupp.) to the parallelogram BC; and FE is ſome other *parallelogram*; therefore (by 7. 5.) it is, as the *parallelogram* AB is to the *parallelogram* FE ſo *is* the *parallelogram* BC to the *parallelogram* FE: but as the *parallelogram* AB is to the *parallelogram* FE ſo (by 1. 6.) is DB to BE: and as the *parallelogram* BC is to the *parallelogram* FE ſo *is* GB to BF; wherefore alſo (by 11. 5.) as DB *is* to BE ſo is GB to BF; wherefore the ſides of the parallelograms AB, BC, *which are* about the equal angles, are reciprocally proportional.

But let the ſides about the equal angles be reciprocally proporportional, and let it be as DB *is* to BE ſo *is* GB to BF: I ſay that the parallelogram AB is equal to the parallelogram BC.

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Book VI.

For becauſe it is, as DB *is* to BE ſo *is* GB to BF; but as DB *is* to BE ſo (by 1. 6.) *is* the parallelogram AB to the parallelogram FE; and as GB *is* to BF ſo *is* the parallelogram BC to the *parallelogram* FE; and therefore (by 11. 5.) as the *parallelogram* AB is to the *parallelogram* FE ſo is the *parallelogram* BC to the *parallelogram* FE: therefore (by 9. 5.) the parallelogram AB is equal to the parallelogram BC.

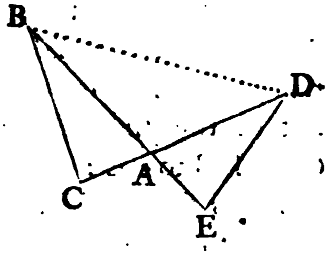
Wherefore the ſides about the equal angles of equal parallelograms which have one angle equal to one angle, are reciprocally proportional: and thoſe parallelograms, having one angle equal to one angle, of which the ſides about the equal angles are reciprocally proportional, are equal. Which was to be demonſtrated.

PROP. XV.

The ſides, about the equal angles of equal triangles, which have one angle equal to one angle, are reciprocally proportional: and thoſe *triangles* having one angle equal to one angle, of which the ſides about the equal angles are reciprocally proportional, are equal.

Let ABC, ADE be equal triangles, having one angle equal to one *angle*; viz. *BAC* equal to DAE: I ſay that the ſides about the equal angles of the triangles ABC, ADE are reciprocally proportional; that is, as CA is to AD ſo *is* EA to AB.

For let the *triangles* be placed in ſuch a manner, that CA, AD may be in a ſtraight line: therefore alſo (by 14. 1.) EA, AB are in a ſtraight line: and let BD be joined.



Then becauſe (by ſupp.) the triangle ABC is equal to the triangle ADE, and ABD is another *triangle*; therefore it is, as the triangle CAB is to the triangle BAD ſo is the triangle ADE to the triangle BAD: but (by 1. 6.) as the *triangle* CAB *is* to BAD ſo *is* CA to AD; and as the triangle EAD is to the *triangle* BAD ſo *is* EA to AB: therefore alſo (by 11. 5.) as CA *is* to AD ſo *is* EA to AB: therefore the ſides about the equal angles of the triangles ABC, ADE are reciprocally proportional.

But let the ſides of the triangles ABC, ADE be reciprocally proportional; and let it be as CA *is* to AD ſo *is* EA to AB: I ſay that the triangle ABC is equal to the triangle ADE.

For again BD being joined: becauſe it is (by ſupp.) as CA is to AD ſo *is* EA to AB; but as CA is to AD (by 1. 6.) ſo *is* the triangle ABC to the triangle BAD; and as EA *is* to AB ſo is the triangle EAD to the triangle BAD: wherefore (by 11. 5.) as the triangle ABC is to the *triangle* BAD ſo is the triangle EAD to the *triangle* BAD: therefore each of the *triangles* ABC, ADE have the ſame ratio to BAD; therefore (by 9. 5.) the triangle ABC is equal to the triangle EAD.

Wherefore the ſides about the equal angles of equal triangles, which have one angle equal to one angle, are reciprocally proportional: and thoſe *triangles* having one angle equal to one angle, of which the ſides about the equal angles are reciprocally proportional, are equal. Which was to be demonſtrated.

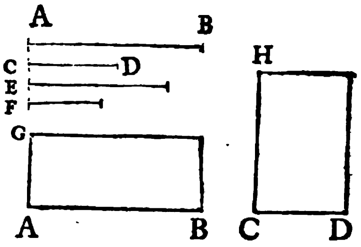
PROP. XVI.

If four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines will be proportionals.

Let the four ſtraight lines AB, CD, E, F be proportionals, viz. as AB *is* to CD ſo *let* E *be* to F: I ſay that the rectangle contained by AB *and* F is equal to the rectangle contained by CD *and* E.

For from the points A, C let AG, CH be drawn at right angles to the ſtraight lines AB, CD; and let AG be made equal to F; and CH equal to E; and let the parallelograms BG, DH be compleated.

And becauſe it is, as AB *is* to CD ſo *is* E to F; and CH *is* equal to E and AG to F: therefore it is (by 7. 5.) as AB *is* to CD ſo *is* CH to AG; therefore the ſides about the equal angles of the parallelograms BG, DH are reciprocally porportional; but thoſe equi-



angular parallelograms are equal (by 14. 6.) of which the ſides about the equal angles are reciprocally proportional: therefore the parallelogram BG is equal to the parallelogram DH: and the *parallelogram* BG is the *rectangle contained* by AB and F; for AG is equal to F: and the *parallelogram* DH is the rectangle contained by CD *and* E; for CH is equal to E: wherefore the rectangle contained by AB *and* F is equal to the rectangle contained by CD *and* E.

But let the rectangle contained by AB *and* F be equal to the rectangle contained by CD *and* E: I ſay that the four ſtraight lines will be proportionals, viz. as AB *is* to CD ſo *will* E *be* to F. For the ſame things being conſtructed; becauſe (by the ſupp.) the *rectangle contained* by AB, F is equal to the *rectangle contained* by CD, E; and the *parallelogram* BG is the *rectangle contained* by AB, F; for AG is equal to F: and the *parallelogram* DH is the rectangle contained by CD, E; for CH is equal to E: therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular: but (by 14. 6.) the ſides about the equal angles of equal and equiangular parallelograms are reciprocally proportional: wherefore it is, as AB is to CD ſo is CH to AG: but CH is equal to E; and AG to F: therefore it is as AB *is* to CD ſo *is* E to F.

Wherefore if four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines will be proportionals. Which was to be demonſtrated.

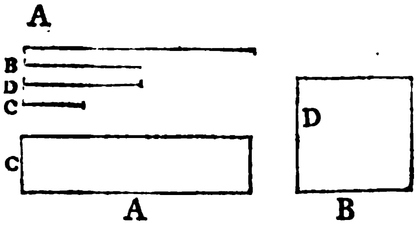
PROP. XVII.

If three ſtraight lines be proportionals; the rectangle contained by the extremes is equal to the ſquare of the mean *proportional*: and if the rectangle contained by the extremes be equal to the ſquare of the mean *term*, the three ſtraight lines will be proportionals.

Let the three ſtraight lines A, B, C be proportionals; viz. as A *is* to B ſo *let* B *be* to C: I ſay that the rectangle contained by A *and* C is equal to the ſquare of the mean B.

Let D be made equal to B.

And becauſe it is, as A *is* to B ſo *is* B to C; and B is equal to D; therefore it is, as A is to B ſo *is* D to C: but if four ſtraight lines be proportionals, the rectangle contained by the extremes is equal (by 16. 6.) to the rectangle contained by the means; therefore the rectangle contained by A *and* C is equal to the rectangle contained by B *and* D: but the rectangle contained by B *and* D is equal to the *ſquare* of B; for B is equal to D: therefore the rectangle contained by A *and* C is equal to the *ſquare* of B.



But let the *rectangle contained* by A and C be equal to the *ſquare* of B; I ſay that it is, as A *is* to B ſo *is* B to C.

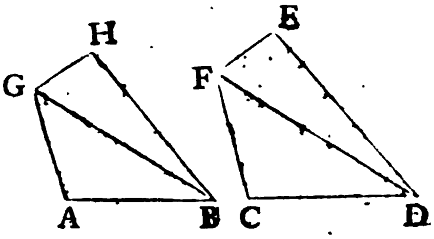
For the ſame things being conſtructed; becauſe (by ſupp.) the rectangle contained by A *and* C is equal to the *ſquare* of B; but the *ſquare* of B is equal to the *rectangle contained* by B *and* D; for B is equal to D: therefore the *rectangle contained* by A *and* C is equal to the *rectangle contained* by B *and* D; but (by 16. 6.) if the *rectangle contained* by the extremes be equal to the rectangle contained by the means; the four ſtraight lines are proportionals: therefore it is, as A *is* to B ſo *is* D to C; but B is equal to D; therefore as A *is* to B ſo *is* B to C.

Wherefore if three ſtraight lines be proportionals; the rectangle contained by the extremes is equal to the ſquare of the mean *proportional*: and if the rectangle contained by the extremes be equal to the ſquare of the mean *term*; the three ſtraight lines will be proportionals. Which was to be demonſtrated.

PROP. XVIII.

Upon a given ſtraight line to deſcribe a rectilineal *figure*, ſimilar and ſimilarly ſituated to a given rectilineal *figure*.

Let AB be the given ſtraight line; and CE the given rectilineal *figure*: it is required, upon the given ſtraight line AB to deſcribe a rectilineal figure ſimilar and ſimilarly ſituated to the rectilineal figure CE.



Let DF be joined; and with the ſtraight line AB and at the points A, B in it, let the *angle* GAB be made equal to the angle at C; and the *angle* ABG *equal* to the *angle* CDF; therefore (by the 32. 1.) the remaining *angle* CFD is equal to the remaining *angle* AGB; therefore the triangle FCD is equiangular to the triangle GAB: therefore there is *this* proportion (by 4. 6.) as FD *is* to GB ſo *is* FC to GA; and ſo *is* CD to AB: Again, with the ſtraight line BG, and at the points B, G in it, let the *angle* BGH be made equal to the angle DFE; and the *angle* GBH equal to the *angle* FDE; therefore the remaining *angle* at H is equal to the remaining *angle* at E: therefore the triangle FDE is equiangular to the triangle GBH; therefore there is this proportion (by 4. 6.) as FD *is* to GB ſo *is* FE to GH; and *ſo is* ED to HB: But it has been alſo demonſtrated *that* as FD is to GB ſo is FC to GA, and *ſo is* CD to AB: therefore (by 11. 5.) as FC is to GA ſo *is* CD to AB and *ſo is* FE to GH, and beſides *ſo is* ED to HB: And becauſe the angle CFD is equal to the angle AGB and DFE *is equal* to BGH; therefore the whole *angle* CFE is equal to the whole AGH. Certainly for the ſame reaſon alſo the *angle* CDE is equal to ABH: but alſo the *angle* at C is equal to the *angle* at A; and the *angle* at E to the *angle* at H: therefore the *figure* AH is equiangular to the *figure* CE; and has the ſides about the equal angles in it proportionals; therefore (by 1. def. 6.) the rectilineal *figure* AH is ſimilar to the rectilineal *figure* CE.

Wherefore upon a given ſtraight line AB a rectilineal *figure* AH hath been deſcribed, ſimilar and ſimilarly ſituated to the given rectilineal *figure* CE. Which was to be done.

PROP. XIX.

Similar triangles are to one another in the duplicate ratio of the ſides of like ratio.

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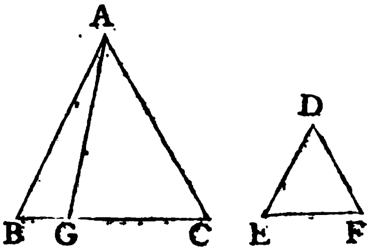
Let ABC, DEF be ſimilar triangles, having the angle at B equal to the *angle* at E: and as AB *is* to BC ſo *let* DE *be* to EF; ſo that (by 12. def. 5.) BC may be the ſide of like ratio to EF: I ſay that the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF.

For (by 11. 6.) let BG be taken a third proportional to BC, EF; ſo that it may be, as BC is to EF ſo *is* EF to BG: and let GA be joined,



Wherefore becauſe it is, as AB *is* to BC ſo *is* DE to EF; therefore alternately (by 16. 5.) it is, as AB *is* to DE ſo *is* BC to EF: but (by conſt.) as BC *is* to EF ſo *is* EF to BG: therefore alſo (by 11. 5.) as AB *is* to DE ſo *is* EF BG: therefore the ſides about the equal angles of the triangles ABG, DEF are reciprocally proportional: but thoſe *triangles* having one angle equal to one angle, and of which the ſides about the equal angles are reciprocally proportional, are equal. Therefore the triangle ABG is equal to the triangle DEF: and becauſe it is, as BC *is* to EF ſo *is* EF to BG; and if three ſtraight lines be proportionals (by 10. def. 5.) the firſt is ſaid to have to the third a duplicate ratio of that which it *has* to the ſecond: therefore BC has to BG a duplicate ratio of that which BC *has* to EF: but as BC *is* to BG ſo (by 1. 6.) *is* the triangle ABC to the triangle ABG: therefore alſo the triangle ABC has to the triangle ABG the duplicate ratio of that which BC *has* to EF: but the triangle ABG is equal to the triangle DEF; therefore (by 7. 5.) the triangle ABC has to the triangle DEF the duplicate ratio of that which BC *has* to EF.

Wherefore ſimilar triangles are to one another in the duplicate ratio of the ſides of like ratio. Which was to be demonſtrated.

Cor. Certainly from this it is manifeſt, that if three ſtraight lines be proportionals, it is, as the firſt *is* to the third ſo *is* a triangle deſcribed upon the firſt, to a ſimilar and ſimilarly ſituated *triangle deſcribed* upon the ſecond; ſince it has been demonſtrated, as CB *is* to BG ſo *is* the triangle ABC to the triangle ABG, that is DEF.

PROP. XX.

Similar polygons are divided into ſimilar triangles, and into the ſame number, and *have* the ſame ratio *to one another which* the whole *polygons have*: and the *one* polygon has to the *other* polygon the duplicate ratio of that which one ſide of the one has to the ſide of like ratio of the other.

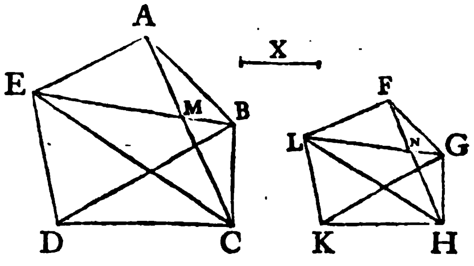
Let ABCDE, FGHKL be ſimilar polygons, and let AB be the ſide of like ratio FG: I ſay that the polygons ABCDE, FGHKL are divided into ſimilar triangles; and into the ſame number; and have the ſame ratio *to one another which* the whole *polygons have*; and *that* the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB *has* to FG.

Let BE, EC; GL, LH be joined.

And becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE is equal to the *angle* GFL: and it is, as BA *is* to AE ſo *is* GF to FL: wherefore becauſe there are two triangles ABE, FGL having one angle equal to one angle, and the ſides about the equal angles proportionals; therefore (by 6. 6.) the triangle ABE is equiangular to the triangle FGL; ſo that (by 4. 6.) it is alſo ſimilar: therefore the angle ABE is equal to the angle FGL: and alſo on account of the ſimilarity of the polygons, the whole *angle* ABC is equal to the whole *angle* FGH; therefore the remaining angle EBC is equal to the remaining *angle* LGH; and becauſe, on account of the ſimilarity of the triangles ABE, FGL, it is, as EB *is* to BA ſo *is* LG to GF; but alſo, on account of the ſimilarity of the polygons, it is, as AB *is* to BC ſo *is* FG to GH: therefore by equality it is, (by 22, 5.) as EB *is* to BC ſo *is* LG to GH; and the ſides *therefore* about the equal angles EBC, LGH are proportionals; therefore the triangle EBC is equiangular to the triangle LGH (by 6. 6.); ſo that *it is* alſo ſimilar (by 4. 6.): Certainly for the ſame reaſon the triangle ECD is ſimilar to the triangle LHK: therefore the ſimilar polygons ABCDE, FGHKL are divided into ſimilar triangles, and into an equal number.

I ſay alſo that they have the ſame ratio to one another which the whole polygons have: that is, ſo that the triangles are proportionals; and that the antecedents are the *triangles* ABE, EBC, ECD and the conſequents of them, the *triangles* FGL, LGH, LHK: alſo that the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide of like ratio *of the one* has to the ſide of like ratio *of the other*, that is, which AB *has* to FG.

For let AC, FH be joined.



And becauſe, on account of the ſimilarity of the polygons, the angle ABC is equal to the *angle* FGH, and it is as AB *is* to BC ſo is FG to GH; the triangle ABC is equiangular (by 6. 6.) to the triangle FGH: therefore the *angle* BAC is equal to the *angle* GFH; and BCA to GHF: and becauſe the angle BAM is equal to GFN; and the angle ABM has been demonſtrated *to be* equal to FGN; therefore the remaining *angle* AMB is equal (by 32. 1.) to the remaining *angle* FNG; therefore the triangle ABM is equiangular to the triangle FGN: Certainly in the ſame manner we ſhall demonſtrate that the *triangle* BMC is equiangular to the triangle GNH; therefore there is this proportion, (by 4. 6.) as AM *is* to MB ſo *is* FN to NG; and as BM *is* to MC ſo *is* GN to NH: ſo that alſo by equality (by 22. 5.) as AM is to MC ſo *is* FN to NH: but as AM *is* to MC ſo (by 1. 6.) *is* the triangle ABM to the *triangle* BMC; and *ſo is the triangle* AEM to the *triangle* MEC; for they are to one another as *their* baſes: And (by 12. 5.) as one of the antecedents is to one of the conſequents ſo *are* all the antecedents to all the conſequents; therefore as the triangle AMB is to the triangle BMC ſo *is* the *triangle* ABE to the *triangle* CBE: but as the *triangle* AMB *is* to the *triangle* BMC ſo is AM to MC: therefore alſo (by 11. 5.) as AM *is* to MC ſo *is* the triangle ABE to the triangle CBE. Certainly for the ſame reaſon alſo as FN is to NH ſo is the triangle FGL to the triangle GLH: and it is, as AM *is* to MC ſo *is* FN to NH; therefore (by 11. 5.) as the triangle ABE *is* to the triangle BEC ſo *is* the triangle FGL to the triangle GHL; and alternately (by 16. 5.) as the triangle ABE *is* to the triangle FGL ſo *is* the triangle BEC to the triangle GLH.

Certainly in the ſame manner we ſhall demonſtrate, BD and GK being joined, that alſo, as the triangle BEC is to the triangle GLH ſo is the triangle ECD to the triangle LHK: and becauſe it is, as the triangle ABE *is* to the triangle FGL ſo is the triangle EBC to the triangle LGH, and beſides ſo *is* the triangle ECD to the triangle LHK: wherefore alſo (by 12. 5.) as one of the antecedents is to one of the conſequents, ſo *are* all the antecedents to all the conſequents; therefore as the triangle ABE is to the triangle FGL ſo is the polygon ABCDE to the polygon FGHKL: but the triangle ABE has to the triangle FGL (by 19. 6.) the duplicate ratio of that which the ſide of like ratio AB has to the ſide of like ratio FG; for ſimilar triangles are in the duplicate ratio of the ſides of like ratio; therefore alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide of like ratio AB *has* to the ſide of like ratio FG.

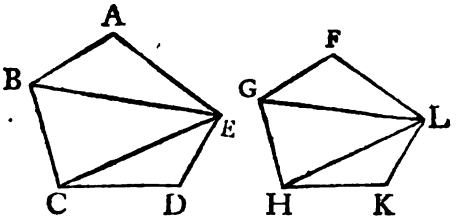
Wherefore ſimilar polygons are divided into ſimilar triangles, and into an equal number of them, and of like ratio with the whole polygons; and the *one* polygon has to the *other* polygon the duplicate ratio of that which *one* ſide of like ratio has to another ſide of like ratio. Which was to be demonſtrated.

Cor. 1. Certainly in like manner it will be demonſtrated in quadrilateral figures, that they are in the duplicate ratio of the ſides of like ratio; and it has been demonſtrated alſo in triangles; ſo that univerſally ſimilar rectilineal figures are to one another in the duplicate ratio of the ſides of like ratio.

Cor. 2. And if we take the *line* X (by 11. 6.) a third proportional to AB, FG; (by 10. def. 5.) AB has to X the duplicate ratio of that which AB *has* to FG: and alſo a polygon to a polygon, and a quadrilateral *figure* to a quadrilateral *figure* has the duplicate ratio of that which the ſide of like ratio has to the ſide of like ratio, that is, *which* AB *has* to FG: and this has been demonſtrated alſo in triangles; ſo that it *is* alſo manifeſt univerſally, that if three ſtraight lines be proportionals, it will be, as the firſt *is* to the third ſo is a *rectilineal* figure deſcribed upon the firſt to a ſimilar and ſimilarly ſituated *one deſcribed* upon the ſecond.

OTHERWISE. We ſhall demonſtrate in a different manner and more expeditiouſly, that the triangles are of like ratio.

For again let the polygons ABCDE, FGHKL be ſuppoſed to be deſcribed; and let BE, EC; GL, LH be joined: I ſay that it is, as the triangle ABE *is* to the triangle FGL ſo *is* the *triangle* EBC to the *triangle* LGH, and CDE to HKL.

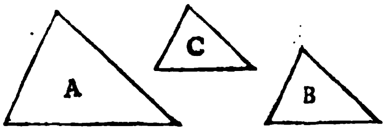


For becauſe the triangle ABE is ſimilar to the triangle FGL; therefore (by 19. 6.) the triangle ABE has to the triangle FGL the duplicate ratio of that which BE *has* to GL. Certainly for the ſame reaſon alſo the triangle BEC has to the triangle GLH the duplicate ratio of that which BE has to GL: therefore it is (by 11. 5.) as the triangle ABE is to the *triangle* FGL ſo is the *triangle* EBC to the *triangle* LGH. Again, becauſe the triangle EBC is ſimilar to the triangle LGH; therefore EBC has to LGH the duplicate ratio of that which the ſtraight line CE has to the *ſtraight line* HL. Certainly for the ſame reaſon alſo, the triangle ECD has to the triangle LHK the duplicate ratio of that which CE *has* to HL; therefore it is, (by 11. 5.) as the triangle EBC *is* to the *triangle* LGH ſo *is* the *triangle* ECD to the *triangle* LHK: but it has been demonſtrated *that* as the *triangle* EBC *is* to LGH ſo is ABE to FGL; therefore alſo as ABE *is* to FGL ſo *is* BEC to LGH, and ſo *is* ECD to LHK: wherefore alſo (by 12. 5.) as one of the antecedents is to one of the conſequents ſo are all the antecedents to all the conſequents; and the remainder as in the former demonſtration. Which was to be demonſtrated.

PROP. XXI.

Figures which are ſimilar to the ſame rectilineal figure; are alſo ſimilar to one another.

For let each of the rectilineal figures A, B be ſimilar to C: I ſay that A is alſo ſimilar to B.



For becauſe A is ſimilar to C, it is alſo equiangular to it, and has the ſides about the equal angles proportionals (by 1. def. 6.). Again becauſe B is ſimilar to C; it is alſo equiangular to it, and has the ſides about the equal angles proportionals: therefore each of the *figures* A, B are equiangular to C and have the ſides about *their* equal angles proportionals *to the ſides about the equal angles of the figure C*; wherefore the *figure* A is equiangular to B (by com.not.1.) and has (by 11. 5.) the ſides about the equal angles proportionals; wherefore (by 1. def. 6.) *the figure* A is ſimilar to the *figure* B. Which was to be demonſtrated.

PROP. XXII.

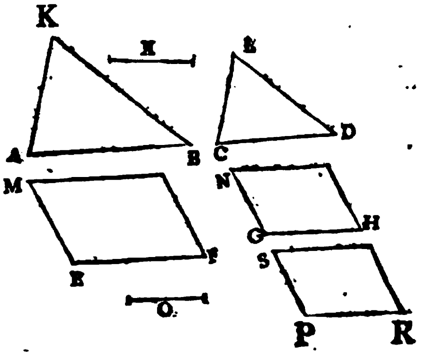
If four ſtraight lines be proportionals; alſo the rectilineal figures ſimilar and ſimilarly deſcribed upon them will be proportionals: and if the rectilineal figures ſimilar and ſimilarly deſcribed upon them be proportionals; alſo the ſtraight lines themſelves will be proportionals.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. as AB *is* to CD ſo *let* EF *be* to GH; and let the rectilineal figures KAB, LCD be deſcribed upon AB, CD ſimilar and ſimilarly ſituated; and the rectilineal figures MF, NH upon EF, GH ſimilar and ſimilarly ſituated: I ſay that it is, as KAB *is* to LCD ſo *is* MF to NH.

For let X be taken (by 11. 6.) a third proportional to AB, CD: and O a third proportional to EF, GH: and becauſe it is, as AB *is* to CD ſo *is* EF to GH; and as CD *is* to X ſo *is* GH to O; therefore by equality it is, (by 22. 5.) as AB is to X ſo *is* EF to O: But as AB is to X ſo (by cor. 2. to 20. 6.) is the *figure* KAB to LCD; and as EF is to O ſo is the *figure* MF to NH; therefore (by 11. 5.) as the *figure* KAB is to LCD ſo is the *figure* MF to NH.

But let it be, as the figure KAB is to LCD ſo *let* the figure MF *be* to NH: I ſay that it is, as AB *is* to CD ſo *is* EF to GH.

For as AB is to CD ſo let EF be made (by 12. 6.) to PR; and let the rectilineal figure SR be deſcribed (by 18. 6.) upon PR ſimilar and ſimilarly ſituated to either of the figures MF, NH.



Wherefore becauſe it is, as AB *is* to CD ſo *is* EF to PR; and the *figures* KAB, LCD have been deſcribed upon AB *and* CD ſimilar and ſimilarly ſituated; and the figures MF *and* SR ſimilar and ſimilarly ſituated upon EF *and* PR; therefore it is (by part. 1. of this.) as the *figure* KAB is to LCD ſo *is* the *figure* MF to SR; but it is alſo ſuppoſed as the *figure* KAB *is* to LCD ſo *is* the *figure* MF to NH; therefore (by 11. 5.) the *figure* MF has the ſame ratio to each of the *figures* NH and SR; therefore (by 9. 5.) NH is equal to SR; but it is ſimilar to it and ſimilarly ſituated: therefore (by the following lemma) GH is equal to PR; and becauſe it is, as AB is to CD ſo is EF to PR (by conſt.): and PR is equal to GH; therefore it is, as AB *is* to CD ſo *is* (by 7. 5.) EF to GH.

Wherefore if four ſtraight lines be proportionals; alſo the ſimilar rectilineal figures ſimilarly deſcribed upon them will be proportionals: and if the ſimilar figures ſimilarly deſcribed upon them be proportionals; alſo the ſtraight lines themſelves will be proportionals. Which was to be demonſtrated.

LEMMA. If rectilineal figures be equal and ſimilar, we ſhall thus demonſtrate, that the ſides of them, which are of like ratio, are equal.

Let NH and SR be equal and ſimilar rectilineal figures; and let it be, as HG *is* to GN ſo *is* RP to PS: I ſay that PR is equal to HG.

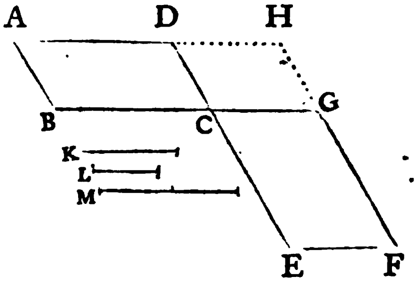
For if they are unequal, one of them is greater; let RP be greater than HG: and becauſe it is, as RP *is* to PS ſo *is* HG to GN, and alternately, as RP is to HG ſo is PS to GN: but PR is greater than GH; therefore alſo PS is greater than GN: ſo that alſo (by 20. 6.) the *figure* RS is greater than the *figure* HN; but it is alſo equal, which is impoſſible; therefore PR is not unequal to GH; therefore equal. Which was to be demonſtrated.

PROP. XXIII.

Equiangular parallelograms have to one another the ratio compounded *of the ratios* of their ſides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the *angle* ECG: I ſay that the parallelogram AC has to the parallelogram CF the ratio compounded *of the ratios* of the ſides; of that which BC has to CG; and of that which DC has to CE.

For let them be placed ſo that BC may be in a ſtraight line with CG; therefore (by 14. 1.) DC is in a ſtraight line with CE: and let the parallelogram DG be compleated; and let K any ſtraight line be drawn; and, as BC *is* to CG ſo (by 12. 6.) let K be made to L; and as DC is to CE ſo *let* L *be made* to M.



Therefore (by conſt.) the ratios of K to L and of L to M, are the ſame *ratios* with the ratios of the ſides; viz. of BC to CG and of DC to CE: but the ratio of K to M is compounded both of the ratio of K to L and *of the ratio* of L, to M; ſo that alſo K has to M the ratio compounded of *the ratios* of the ſides.

And becauſe it is (by 1. 6.) as BC *is* to CG ſo *is* the parallelogram AC to the *parallelogram* CH: but as BC *is* to CG ſo (by conſt.) *is* K to L; therefore alſo (by 11. 5.) as K is to L ſo is the *parallelogram* AC to the *parallelogram* CH: Again becauſe it is, (by 1. 6.) as DC is to CE ſo is the parallelogram CH to the *parallelogram* CF: but as DC is to CE ſo (by conſt.) is L to M; therefore alſo (by 11. 5.) as L is to M ſo is the *parallelogram* CH to the *parallelogram* CF: wherefore becauſe it has been demonſtrated, *that* as K is to L ſo is the parallelogram AC to the parallelogram CH; and as L is to M ſo *is* the parallelogram CH to the *parallelogram* CF; therefore by equality it is, (by 22. 5.) as K is to M ſo is the parallelogram AC to the *parallelogram* CF; and K has to M the ratio compounded *of the ratios* of the ſides; therefore alſo, the *parallelogram* AC has to the *parallelogram* CF the ratio compounded *of the ratios* of the ſides.

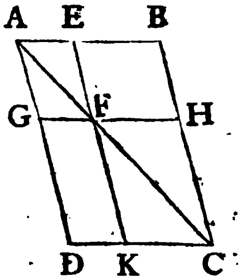
Wherefore equiangular parallelograms have to one another the ratio compounded *of the ratios* of their ſides. Which was to be demonſtrated.

PROP.

PROP. XXIV.

The parallelograms, which are about the diameter of every parallelogram, are ſimilar to the whole and to one another.

Let ABCD be a parallelogram; and AC its diameter; and let EG, HK be parallelograms about the *diameter* AC: I ſay that each of the parallelograms EG, HK is ſimilar to the whole *parallelogram* ABCD and to one another.



For becauſe EF has been drawn parallel to BC one of the ſides of the triangle ABC, there is this proportion (by 2. 6.), as BE *is* to EA ſo *is* CF to FA: again becauſe FG hath been drawn parallel to CD one of the ſides of the triangle ADC; therefore there is this proportion, as CF *is* to FA ſo *is* DG to GA: but as CF is to FA ſo has BE been demonſtrated *to be* to EA: therefore alſo (by 11. 5.) as BE is to EA ſo is DG to GA; and by compoſition therefore (by 18. 5.) as BA *is* to AE ſo *is* DA to AG; and alternately (by 16. 5.) as BA *is* to AD ſo *is* EA to AG: therefore the ſides about the common angle BAD of the parallelograms ABCD *and* EG are proportionals. And becauſe GF is parallel to DC, the angle AGF (by 29. 1.) is equal to ADC; and the angle GFA to DCA; and the angle DAC is common to the two triangles ADC, AGF; therefore the triangle ADC is equiangular to the triangle AGF. Certainly for the ſame reaſon alſo the triangle ABC is equiangular to the triangle AFE: therefore alſo the whole parallelogram ABCD is equiangular to the parallelogram EG: therefore there is this proportion, (by 4. 6.) as AD *is* to DC ſo *is* AG to GF; and as DC *is* to CA ſo *is* GF to FA; and as AC *is* to CB ſo *is* AF to FE and beſides as CB *is* to BA ſo *is* FE to EA: and becauſe it has been demonſtrated *that* as DC *is* to CA ſo *is* GF to FA; and as AC *is* to CB ſo *is* AF to FE; therefore by equality it is, (by 22. 5.) as DC is to CB ſo is GF to FE: therefore the ſides about the equal angles of the parallelograms ABCD and EG are proportionals: therefore (by 1. def. 6.) the parallelogram ABCD is ſimilar to the parallelogram EG. Certainly for the ſame reaſon alſo the parallelogram ABCD is ſimilar to the parallelogram HK;

therefore each of the parallelograms EG, HK is ſimilar to the parallelogram ABCD: but (by 21. 6.) *figures* which are ſimilar to the ſame rectilineal *figure* are alſo ſimilar to one another: therefore alſo the parallelogram EG is ſimilar to the parallelogram HK.

Wherefore the parallelograms which are about the diameter of every parallelogram are ſimilar to the whole, and to one another. Which was to be demonſtrated.

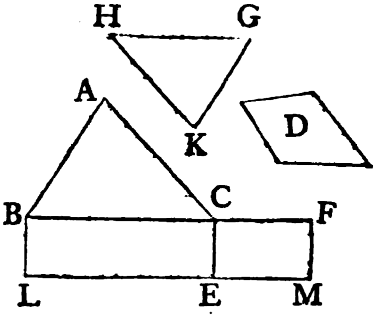
PROP. XXV.

To make the ſame *rectilineal figure* ſimilar to a given rectilineal figure, and equal to another given *rectilineal figure*.

Let ABC be the given rectilineal figure, to which it is required that the *figure* ſhould be made ſimilar; and D the *one* to which *it is to be* equal: it is required that the ſame figure be made ſimilar to ABC and equal to D.

For let the parallelogram BE be applied to the ſtraight line BC (by 44. 1.) equal to the triangle ABC; and (by 45. 1.) *let* the parallelogram CM *be applied* to the ſtraight line CE, equal to the *figure* D, in the angle FCE which is equal to the *angle* CBL; therefore BC is (by 14. 1.) in a ſtraight line with CF; and LE with EM; and let GH be taken (by 13. 6.) a mean proportianal between BC, CF: and let the *figure* KGH be deſcribed (by 18. 6.) upon GH ſimilar to ABC, and ſimilarly ſituated.

And becauſe it is, as BC *is* to GH ſo *is* GH to CF (by conſt.); and (by 2. Cor. to 20. 6.) if three ſtraight lines be proportionals, it is, as the firſt *is* to the third ſo *is* the *rectilineal* figure upon the firſt to a ſimilar and ſimilarly deſcribed *figure* upon the ſecond; therefore it is, as BC *is* to CF ſo *is* the triangle ABC to KGH; but alſo (by 1. 6.) as BC *is* to CF ſo *is* the parallelogram BE to the parallelogram EF; therefore alſo (by 11. 5.) as the triangle ABC *is* to the triangle KGH ſo *is* the parallelogram BE to the parallelogram EF: alternately (by 16. 5.) there-



therefore as the triangle ABC *is* to the parallelogram BE ſo *is* the triangle KGH to the parallelogram EF: but (by conſt.) the triangle ABC *is* equal to the parallelogram BE; therefore alſo the triangle KGH *is* equal to the parallelogram EF: but the parallelogram EF is equal to the *rectilineal figure* D: therefore alſo KGH is equal to D; and KGH is alſo ſimilar to ABC.

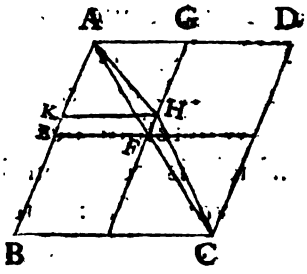
Wherefore, the ſame *figure* KGH has been made ſimilar to the given rectilineal *figure* ABC and equal to another given figure. Which was to be done.

PROP. XXVI.

If a parallelogram be taken away from a parallelogram ſimilar to the whole and ſimilarly ſituated, having a common angle with it; it is about the ſame diameter with the whole *parallelogram*.

For let the parallelogram AEFG be taken away from the parallelogram ABCD, ſimilar to ABCD, and ſimilarly ſituated, having the angle DAB common with it: I ſay that ABCD is about the ſame diameter with AEFG.

For *if* not, but if poſſible, let AHC be the diameter [of them] *of the parallelogram BD*; and let HK be drawn through the point H parallel to either of the *lines* AD, BC.

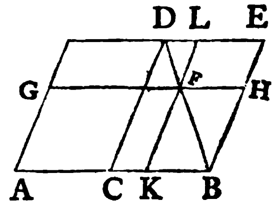


Wherefore becauſe ABCD is about the ſame diameter with the *parallelogram* KG; ABCD is (by 24. 6.) ſimilar to the *parallelogram* KG; therefore it is (by 1. def. 6.) as DA *is* to AB ſo *is* GA to AK; but it is alſo, on account of the ſimilarity of the *parallelograms* ABCD, EG as DA *is* to AB ſo *is* GA to AE; therefore alſo (by 11. 5.) as GA is to AE ſo is GA to AK: therefore GA has the ſame ratio to each of the *lines* AE, AK; therefore (by 9. 5.) AE is equal to AK; the leſs to the greater; which is impoſſible: therefore ABCD is not about the ſame diameter with the *parallelogram* KG: therefore the parallelogram ABCD is about the ſame diameter with the parallelogram AEFG.

Wherefore if a parallelogram be taken away from a parallelogram ſimilar to the whole and ſimilarly ſituated, having a common angle with it; it is about the ſame diameter with the whole *parallelogram*. Which was to be demonſtrated.

PROP. XXVII.

Of all the parallelograms applied to the ſame ſtraight line, and deficient by parallelogram figures ſimilar and ſimilarly ſituated to the *parallelogram* applied to half the line: the parallelogram applied to half the line, *as* being ſimilar to the defect, is the greateſt.



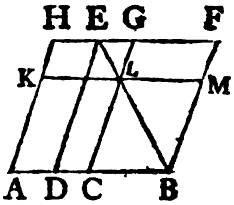
Let AB be a ſtraight line; and let it be cut in halves in the point C: and let the parallelogram AD be applied to the ſtraight line AB; deficient by the parallelogram figure CE, ſimilar and ſimilarly ſituated to that deſcribed upon the half of the *ſtraight line* AB, that is upon BC: I ſay that AD is the greateſt of all the parallelograms applied to the ſtraight line AB, and deficient by parallelogram figures ſimilar and ſimilarly ſituated to CE. For let the parallelogram AF be applied to the ſtraight line AB, deficient by the parallelogram figure KH, ſimilar and ſimilarly ſituated to CE; I ſay that AD is greater than AF.

For becauſe the parallelogram CE is ſimilar to the parallelogram KH, they are (by 26. 6.) about the ſame diameter: let their diameter DB be drawn: and let the figure be deſcribed.

Wherefore becauſe CF is equal (by 43. 1.) to FE, let KH which is common be added, therefore the whole CH is equal to the whole KE: but CH is equal to CG, becauſe AC is equal to CB; therefore alſo CG is equal to EK: let CF which is common be added; therefore the whole AF is equal to the gnomon CHL: ſo that the parallelogram CE, that is AD (by 36. 1.) is greater than the parallelogram AF.

But again let the ſtraight line AB be cut in halves in the point C; and AL being applied deficient by the figure CM; and again let the parallelogram AE be applied to AB deficient by DF ſimilar and ſimilarly ſituated to CM the *parallelogram* applied to half the *line* AB: I ſay that AL the *parallelogram* applied to half the line is greater than AE.

For becauſe DF is ſimilar to GM, they are (by 26. 6.) about the ſame diameter; let their diameter be EB; and let the figure be deſcribed.



And becauſe LF is equal to LH (by 36. 1.) becauſe FG is equal to GH; therefore LF is equal to DL; therefore DL is greater than EK; let DK *which is* common be *added*; therefore the whole AL is greater than the whole AE.

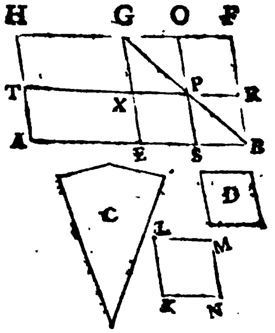
Wherefore of all the parallelograms applied to the ſame ſtraight line, and deficient by parallelogram figures ſimilar and ſimilarly ſituated to the *parallelogram* applied to half the line: the parallelogram applied to half the line, *as* being ſimilar to the defect, is the greateſt. Which was to be demonſtrated.

PROP. XXVIII.

To apply to a given ſtraight line a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram figure, which is ſimilar to a given parallelogram: but it is required, that the given rectilineal figure to which it is required that the one to be applied is to be equal, ſhould not be greater than the parallelogram applied to half the line; theſe being ſimilar, the deficient *parallelogram*, the one applied to half the line and the parallelogram to which the deficient one is required to be ſimilar.

Let AB be the given ſtraight line; and C the given rectilineal figure, to which it is required to apply to AB one equal; not being greater than the parallelogram applied to half the line, the defects being ſimilar; and let D be the *parallelogram* to which the defect is required to be ſimilar: it is required to apply to the given ſtraight line AB a parallelogram equal to the given rectilineal figure C; deficient by a parallelogram figure which is ſimilar to D.

Let AB be cut in halves at the point E; and let EBFG be deſcribed upon EB ſimilar to D, and ſimilarly ſituated: and let the parallelogram AG be compleated: certainly AG is either equal to C, or greater than it, on account of the limitation: if therefore AG be equal to C, what was required hath been done; for the parallelogram AG hath been applied to the given ſtraight line AB, equal to the given rectilineal figure C, deficient by the paral-



lelogram figure EF, which is ſimilar to D: But if not; HE is greater than C: but HE is equal to EF; therefore alſo EF is greater than C: let the *parallelogram* KLMN be made (by 25. 6.) equal to that exceſs by which EF is greater than C, and ſimilar to D, and the ſame ſimilarly ſituated: but D is ſimilar (by conſt.) to EF; therefore KM is alſo ſimilar to EF; therefore let LK be the ſide of like ratio with GE; and LM the *ſide of like ratio* with GF; and becauſe the *parallelogram* EF is equal to *the figure* C and KM together; therefore EF is greater than KM; therefore (by 1. cor. to 20. 6.) GE is greater than KL and GF than LM: Let GX be made equal to KL and GO equal to LM; and let the parallelogram XGOP be compleated; therefore XO is equal and ſimilar to KM: but KM is ſimilar to EF; therefore (by 21. 6.) XO is alſo ſimilar to EF; therefore XO and EF (by 26. 6.) are about the ſame diameter: let GPB be their diameter, and let the figure be compleated.

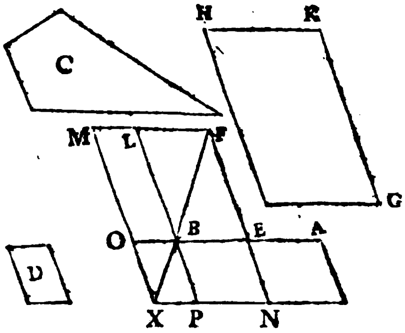
Wherefore becauſe the parallelogram EF is equal to C and KM together; the *parts* of which XO and KM are equal; therefore the remaining gnomon ERO is equal to the remaining *figure* C; and becauſe OR is equal to XS; let RS which is common be added to both; therefore the whole OB is equal to the whole XB: but XB is equal to TE (by 36. 1.) becauſe the ſide AE is equal to the ſide EB; therefore alſo TE is equal to OB; let XS which is common be added; therefore the whole TS is equal to the gnomon ERO: but the gnomon ERO has been demonſtrated to be equal to the *figure* C; therefore alſo TS is equal to the figure C.

Wherefore the parallelogram TS hath been applied to the given ſtraight line AB equal to the given rectilineal figure C, deficient by the parallelogram figure RS which is ſimilar to D; ſince (by 24. 6.) RS is ſimilar to OX. Which was to be done.

PROP. XXIX.

To apply a parallelogram, to a given ſtraight line, equal to a given rectilineal figure, exceeding by a parallelogram figure, ſimilar to a given *parallelogram*.

Let AB be the given ſtraight line; and C the given rectilineal figure to which the *parallelogram* applied to AB is required to be equal: and *let* D *be the parallelogram* to which it is required that the exceeding one be ſimilar: it is required to apply a parallelogram to the ſtraight line AB equal to the rectilineal figure C exceeding by a parallelogram figure ſimilar to D.



Let AB be cut in halves in the *point* E: and let the parallelogram EL be deſcribed upon EB ſimilar to D and ſimilarly ſituated; and let GH be made equal to both, the *parallelogram* EL and the *figure* C together, and alſo ſimilar to D, and the ſame ſimilarly ſituated (by 25. 6.): therefore GH is ſimilar to EL: but let KH be the *ſide* of like ratio with FL; and KG the *ſide of like ratio* with FE: and becauſe the *parallelogram* GH is greater than EL; therefore the *ſide* KH is greater than FL; and KG than FE: Let FL, FE be produced; and let FLM be equal to KH, and FEN equal to GK; and let the *parallelogram* MN be compleated; therefore MN is equal and ſimilar to GH: but GH is alſo ſimilar to EL; therefore alſo (by 21. 6.) MN is ſimilar to EL: therefore (by 26. 6.) EL is about the ſame diameter with MN: let their diameter FX be drawn; and let the figure be deſcribed.

Wherefore becauſe the *parallelogram* GH is equal to EL and C together (by conſt.); but GH is equal to MN, therefore alſo MN is equal to EL and C together: let EL *which is* common be taken away; therefore the remaining gnomon NOL is equal to the figure C: and becauſe EA is equal to EB; the *parallelogram* AN is alſo (by 36. 1.) equal to NB, that is (by 43. 1.) to LO; let EX *which is* common be added; therefore the whole AX is equal to the gnomon NOL; but the gnomon NOL is equal to the *figure* C; therefore AX is alſo equal to the *figure* C.

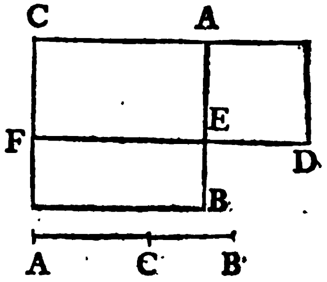
Wherefore the parallelogram AX has been applied to the given ſtraight line AB, equal to the given rectilineal figure C; exceeding by the parallelogram figure PO which is ſimilar to D; becauſe (by 24. 6.) EL is ſimilar to OP. Which was to be done.

PROP. XXX.

To cut a given finite ſtraight line in extreme and mean ratio.

Let AB be the given finite ſtraight line: it is required to cut the ſtraight line AB in extreme and mean ratio.

For let the ſquare BC be deſcribed upon AB; and (by 29. 6.) let the parallelogram CD be applied to AC, equal to the *ſquare* BC, exceeding by the figure AD ſimilar to BC.



But BC is a ſquare, therefore AD is alſo a ſquare: and becauſe BC is equal to CD, let CE *which is* common be taken away; therefore the remainder BF is equal to the remainder AD; and it is alſo equiangular to it; therefore (by 14. 6.) the ſides which are about the equal angles of the *figures* BF, AD are reciprocally proportional; therefore it is, as FE *is* to ED ſo *is* AE to EB; but (by 34. 1.) FE is equal to AC, that is to AB, and ED to AE: therefore it is, as AB *is* to AE ſo *is* AE to EB: But AB is greater than AE; therefore AE is greater than EB.

Wherefore (by 3. def. 6.) the ſtraight line AB is cut in extreme and mean ratio at the point E; and AE is the greater ſegment of it. Which was to be done.

OTHERWISE. Let AB be the given ſtraight line; it required to cut the ſtraight line AB in extreme and mean ratio.

For let AB be cut in C (by 11. 2.) ſo that the rectangle *contained* by AB, BC may be equal to the ſquare of AC.

Wherefore becauſe the rectangle *contained* by AB, BC is equal to the ſquare of AC; therefore it is, (by 17. 6.) as AB *is* to AC ſo *is* AC to CB; therefore AB hath been cut in extreme and mean ratio in the point C. Which was to be done.

PROP. XXXI.

In right angled triangles, the figure *deſcribed* upon the ſide ſubtending the right angle is equal to the figures ſimilar and ſimilarly deſcribed upon the ſides containing the right angle.

Let

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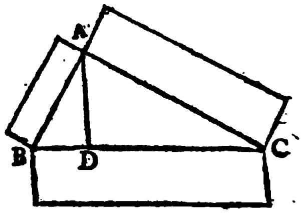
Book VI.

Let ABC be a right angled triangle, having BAC a right angle: I ſay that the figure deſcribed upon BC is equal to the ſimilar and ſimilarly deſcribed figures upon BA, AC.

Let the perpendicular AD be drawn.



Wherefore becauſe, in the right angled triangle ABC, AD hath been drawn from the right angle at A perpendicular to the baſe BC; therefore (by 8. 6.) the triangles ABD, ADC, at the perpendicular, are ſimilar to the whole ABC and to one another: and becauſe ABC is ſimilar to ABD; therefore it is, as CB *is* to BA ſo is AB to BD: and becauſe the three ſtraight lines are proportionals; it is, (by 2. Cor. to 20. 6.) as the firſt *is* to the third ſo *is* the figure upon the firſt to a ſimilar and ſimilarly deſcribed figure upon the ſecond; wherefore as CB *is* to BD ſo *is* the figure upon CB to the ſimilar and ſimilarly deſcribed *figure* upon AB. Certainly for the ſame reaſon alſo as BC is to CD ſo *is* the figure upon CB to the *figure* upon AC: ſo that it is, (by 24. 5.) as BC is to BD, DC ſo is the figure upon BC to the figures ſimilar and ſimilarly deſcribed upon BA, AC: but BC is equal to BD, DC; therefore the figure upon BC is equal to the figures ſimilar and ſimilarly deſcribed upon BA, AC.

Wherefore in right angled triangles, the figure *deſcribed* upon the ſide ſubtending the right angle is equal to the figures ſimilar and ſimilarly deſcribed upon the ſides containing the right angle. Which was to be demonſtrated.

OTHERWISE. Becauſe ſimilar figures are (by 20. 6.) in the duplicate ratio of the ſides of like ratio, therefore the figure upon BC has to the figure upon BA the duplicate ratio of that which BC *has* to BA; but (by Cor. 1. to 20. 6.) the ſquare of BC alſo has to the ſquare of BA the duplicate ratio of that which BC *has* to BA; therefore alſo (by 11. 5.) as the figure upon BC *is* to the *figure* upon BA ſo is the ſquare of BC to the ſquare of BA. Certainly for the ſame reaſon, alſo as the figure upon BC *is* to the figure upon CA ſo *is* the ſquare of BC to the ſquare of CA: ſo that alſo (by 24. 5.) as the figure upon BC is to the figures upon BA, AC ſo is the ſquare of BC to the ſquares of BA, AC: but the ſquare of BC is equal (by 47. 1.) to the ſquares of BA, AC; therefore

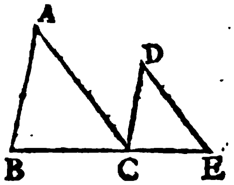
alſo

the figure upon BC is equal to the figures ſimilar and ſimilarly deſcribed upon BA, AC. Which was to be demonſtrated.

PROP. XXXII.

If two triangles, having two ſides proportional to two ſides be joined together at one angle, ſo as that their ſides of like ratio may be alſo parallel; the remaining ſides of the triangles will be in a ſtraight line.

Ket ABC, DCE be two triangles having the two ſides BA, AC proportional to the two ſides CD, DE: ſo that BA *may be* to AC as CD *is* to DE; and AB parallel to CD and AC to DE: I ſay that BC is in a ſtraight line with CE.



For becauſe AB is parallel to DC, and the ſtraight line AC hath fallen upon them, the alternate angles (by 29. 1.) BAC, ACD are equal to one another: Certainly for the ſame reaſon the *angle* CDE is equal to ACD: ſo that alſo (by com. not. 1.) the *angle* BAC is equal to CDE: And becauſe there are two triangles ABC, DCE having one angle the *angle* at A equal to one angle the *angle* at D and the ſides (by ſupp.) about the equal angles proportional; as BA *is* to AC ſo *is* CD to DE; therefore (by 6. 6.) the triangle ABC is equiangular to the triangle DCE: therefore the angle ABC is equal to DCE; and alſo the *angle* ACD has been demonſtrated *to be* equal to BAC: therefore the whole angle ACE is equal to the two angles ABC, BAC; let the common angle ACB be added; therefore the *two* angles ACE, ACB are equal to the *three angles* BAC, ACB, ABC; but (by 32. 1.) the *three angles* BAC, ACB, ABC are equal to two right angles; therefore alſo the angles ACE, ACB are equal to two right angles: with a certain ſtraight line AC and at the point C in it, two ſtraight lines BC, CE not being towards the ſame parts, make the adjacent angles ACE, ACB equal to two right angles; therefore (by 14. 1.) BC is in a ſtraight line with CE.

Wherefore if two triangles, having two ſides proportional to two ſides be joined together at one angle; ſo as that their ſides of like ratio may be parallel: the remaining ſides of the triangles will be in a ſtraight line. Which was to be demonſtrated.

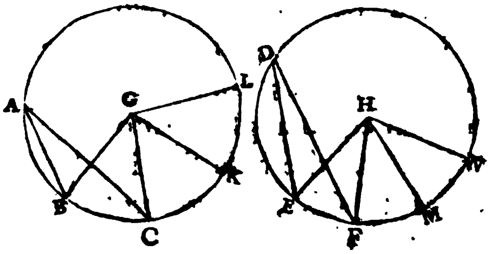
PROP.

PROP. XXXIII.

In equal circles, the angles have the ſame ratio *to one another* as the circumferences on which they ſtand, whether they ſtand at the center or at the circumference. And beſides alſo the ſectors *have the ſame proportion* becauſe they ſtand at the centers.

Let ABC, DEF be equal circles; and let the angles BGC, EHF be at their centers G, H: and the angles BAC, EDF at the circumferences: I ſay that it is, as the circumference BC is to the circumference EF ſo *is* both the angle BGC to the angle EHF and the angle BAC to the angle EDF: and beſides *ſo is* the ſector GBC to the ſector HEF.

Take any number of contiguous *circumferences* (by 1. 4.) CK, KL equal to the circumference BC, and alſo any number *of circumferences which may accidentally happen* FM, MN equal to the circumference EF; and let GK, GL; HM, HN be joined



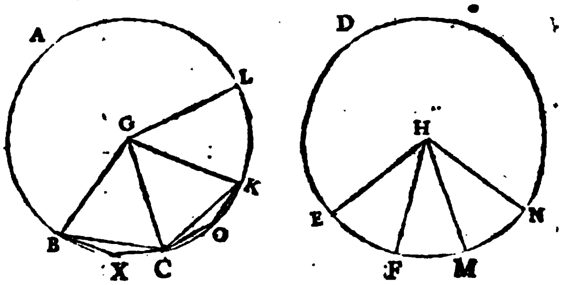
Wherefore becauſe the circumferences BC, CK, KL are equal to one another; alſo (by 27. 3.) the angles BGC, CGK, KGL are equal to one another: whatſoever multiple therefore the circumference BL is of the circumference BC; the ſame multiple alſo is the angle BGL of the angle BGC. Certainly for the ſame reaſon alſo, whatſoever multiple the circumference EN is of the circumference EF; the ſame multiple alſo is the angle EHN of the *angle* EHF: and if the circumference BL be equal to the circumference EN; alſo (by 27. 3.) the angle BGL is equal to the *angle* EHN: and if the circumference BL be greater than the circumference EN; alſo the angle BGL is greater than the angle EHN; and if leſs, leſs: there being four magnitudes; the two circumferences BC, EF; and the two angles BGC, EHF; there have been taken the circumference BL and the angle BGL equimultiples of the circumference BC and of the angle BGC; and the circumference EN and the angle EHN *any other equimultiples which may accidentally happen* of the circumference EF and of the angle EHF; and it has been demonſtrated that if the circumference BL exceed the circumference EN alſo the angle BGL exceeds the angle EHN: and if equal, equal: and if leſs, leſs: it is therefore (by 5. def. 5.) as the circumference BC is to the *circumference* EF ſo is the angle BGC to the *angle* EHF: but as the angle BGC is to the angle EHF ſo is (by 15. 5.) the angle BAC to the angle EDF; for they are (by 20. 3.) the doubles, each of each: therefore as the circumference BC is to the circumference EF ſo *is* both the angle BGC to the angle EHF; and the angle BAC to the angle EDF.

Wherefore in equal circles, the angles have the ſame ratio *to one another* as the circumferences on which they ſtand; whether they ſtand at the centers or at the circumferences. Which was to be demonſtrated.

I ſay alſo, that as the circumference BC *is* to the circumference EF ſo *is* the ſector GBC to the ſector HEF.

For let BC, CK be joined; and the points X, O being taken in the circumferences BC, CK; let BX, XC; CO, OK be joined.

And becauſe the two BG, GC are equal to the two CG, GK; and they contain equal angles; alſo (by 4. 1.) the baſe BC is equal to CK; therefore alſo the triangle BGC is equal to the triangle GCK: and becauſe the circumference BC is equal to the circumference CK; alſo the remaining circumference, which *makes up* the whole circle ABC, is equal to the remaining circumference which *makes up* the ſame circle; ſo that alſo (by 27. 3.) the angle BXC is equal to the angle COK; wherefore (by 11. def. 3.) the ſegment BXC is ſimilar to the ſegment COK; and they are upon equal ſtraight lines BC, CK: but ſimilar ſegments of circles upon equal ſtraight lines (by 24. 3.) are equal to one another: therefore the ſegment BXC is equal to the ſegment COK: but the triangle BGC is alſo equal to the triangle CGK; therefore the whole ſector GBC is equal to the whole ſector GCK. Certainly for the ſame reaſon alſo the ſector GKL is equal to either GKC *or* GCB: therefore the three ſectors GBC, GCK, GKL are equal to one another.



Certainly for the ſame reaſon alſo, the ſectors HEF, HFM, HMN are equal to one another; whatſoever multiple therefore the circumference BL is of the circumference BC; the ſector GBL is alſo the ſame multiple of the ſector GBC. Certainly for the ſame reaſon alſo, whatſoever multiple the circumference EN is of the circumference EF; the ſector HEN is alſo the ſame multiple of the ſector HEF: And if the circumference BL be equal to the circumference EN; alſo the ſector GBL is equal to the ſector HEN: and if the circumference BL exceed the circumference EN; the ſector GBL alſo exceeds the ſector HEN; and if *the one* is deficient, *the other* is deficient. There being then four magnitudes, the two circumferences BC, EF; and the two ſectors GBC, HEF; and equimultiples have been taken of the circumference BC, and of the ſector GBC, viz. the circumference BL and the ſector GBL; and equimultiples alſo of the circumference EF, and of the ſector HEF, viz. the circumference EN and the ſector HEN: and it hath been demonſtrated, that if the circumference BL exceed the circumference EN, the ſector GBL alſo exceeds the ſector HEN; and if equal, equal; and if *the one* is deficient *the other* is deficient. Therefore it is (by 5, def. 5.) as the circumference BC is to the circumference EF ſo is the ſector GBC to the ſector HEF.

Cor. And it is manifeſt (by 11. 5.) alſo, that as the ſector is to the ſector ſo is the angle to the angle.

THE END OF THE SIXTH BOOK.

THE CONCLUSION.

AFTER the firſt ſix books of theſe elements are read, it will be very proper for readers of all denominations, to make ſome kind of eſtimate of their improvement in the knowledge of magnitude, by taking a review of the whole. And this will be done with ſome aſtoniſhment by the judicious reader, when he conſiders to what a hight he has raiſed himſelf above the vulgar in his conceptions upon this ſubject; or indeed even above himſelf, when he compares the common perception which he ſet out with, that *Magnitudes which exactly agree together are equal,* with the power of making a rectilineal figure equal to *any* given rectilineal figure, and ſimilar to *any* other which may be given. But as an admiration of the great abilities of the author, and of his judicious arrangement will be the neceſſary conſequence of right notions upon this ſubject; I ſhall decline all encomiums upon theſe, to turn my thoughts to the aſſiſtance of thoſe who may not have reaſon to be ſo well ſatisfied with their improvement; which is likely to be moſt effectually given, by an enumeration of thoſe circumſtances, to which their want of ſucceſs is moſt probably owing; an attention to which, upon ſome future peruſal, may remove every defect. And although I have mentioned them in the courſe of theſe diſſertations yet it may be uſeful as a concluſion to the whole to leave ſuch an impreſſion upon the reader's mind, as a ſummary of the whole is likely to make. The reader therefore is to conſider firſt of all whether he may not have had all this time, even a wrong notion of the very ſubject. The ſubject is magnitude, and not number; nor is it ſufficient for him to be ſatisfied that *things* equal to the ſame are equal to one another; but he muſt convince himſelf, *not give his aſſent*, that magnitudes which are equal to the ſame, are equal to one another; and this is to be done by a particular examination, of as many different kinds of magnitudes as he can eaſily recollect, eſpecially of ſuch, as are thus compared in theſe firſt ſix books. Next let him examine whether he has a proper notion of a definition, which he muſt do his endeavour to collect both from its nature and uſe; ſtill keeping in his mind that every thing here refers to magnitude. And the ſhewing that any property is conſiſtent with the definition is called a demonſtration. But ſeveral demonſtations ought to be carefully examined by a ſtudent in the circumſtances of the perſon to whom I am now writing, entirely with a view to ſee whether he underſtands what is meant by a geometrical demonſtration; and as we ſuppoſe him to have his choice of the firſt ſix books; he ought eſpecially to pick out the indirect demonſtrations, which are of all others the moſt elegant and forcible, becauſe they convince you contrary to the repreſentation of the figure: the evidence of ſuch propoſitions therefore, if its force be perceived at all, comes to the mind purer, as it is not affected by the prejudices which the reaſoning upon a particular figure, which ſeems to repreſent the conſequences, is but to apt to introduce. For ſuch reaſonings are often diſturbed even by turning the figure upſidedown; and I myſelf have often puzzled a ſtudent ſufficiently acute, by ſuch trials as joining AD in the twenty firſt propoſition of the firſt book, and requiring him to prove that AD and DC together are leſs than AB and BC together &c. The indirect demonſtrations therefore are of all others the propereſt for a ſtudent to exerciſe himſelf in; as this will be the means of removing that very great difficulty which learners find in making a ready and proper uſe of the ſuppoſition; as his chief dependence for conviction in the indirect demonſtrations, muſt be upon the ſuppoſition. It is true this may be got as I have remarked before, by an examination of the ſuppoſition in every propoſition, eſpecially ſuch as conſiſt of ſeveral circumſtances; and varying them, one by one, obſerving the effect which every variation has upon demonſtration; for by ſuch a proceeding the force and uſe of the different parts of the ſuppoſition muſt be diſcovered, and indeed hardly by any other means. And in this progreſs he will be very much aſſiſted by deſcribing his figures will all the variety of poſitions, which his inſtruments and the ſuppoſition will admit of, until he is certain that he has conquered every prejudice which can ariſe from reaſoning upon a particular figure.

The reader is alſo carefully to obſerve the different methods of compariſon made uſe of in the firſt ſix books. In the firſt book, the magnitudes are compared either; as exactly agreeing together; or as being equal to the ſame magnitude; or as being radius's of the ſame circle; which are all common and familiar notions. But in the ſecond book, the method of compariſon is by the rectangle, which is by no means a common notion, either as to its name or nature; and therefore ſhould be made familiar; as well as perfectly underſtood. In ſome part of the third book, the method of compariſon is by ſimilar ſegments, which is alſo an artificial idea, and ought for the ſame reaſon to be made familiar. But the moſt artificial and therefore the moſt difficult of all, is that method of compariſon founded upon a conſideration of quantities as parts and multiples. It will be particularly uſeful for the ſtudent, who has failed in his firſt attempt to underſtand this fifth book, after reading carefully what I have ſaid in explanation of the two firſt definitions, to take care to have the ſuppoſitions given him by another; and then carefully to perform every conſtruction himſelf. And indeed the reader who would take the eaſieſt way of making himſelf maſter of this ſubject, ſhould have the ruler and compaſſes in his hand, through the firſt ſix books, only taking care not to forget, that he is handling a ruler and compaſſes.

THE END OF THE FIRST VOLUME.

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